Corporate Liquidity Management under Moral Hazard

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June 19, 2020

Abstract

We present a dynamic model of liquidity management in which a firm coordinates payouts to both a controlling insider and dispersed outside investors. The insider is subject to moral hazard and the firm faces financing constraints. The insider's performance sensitivity can lie above the minimum required for incentive purposes, serving as a form of risk sharing. Agency problems prevent some firms from attracting initial financing, while others may only do so with risk of future failure. Volatile cash flows exacerbate moral hazard and, therefore, may reduce target cash holdings and force the insider to cover a substantial portion of the initial cash balance. Periodic access to frictionless capital markets allows refinancing and decreases a firm's risk of failure, but does not change the set of firms that can attract initial financing. Agency conflicts lead to state dependent flotation costs as well as cash bonuses to the insider upon refinancing.

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Financial constraints play a key role for the investment, compensation and payout decisions of both firms (Campello et al. 2010) and investment funds, such as private equity and venture capital funds (Robinson and Sensoy 2013). The inability to raise external funds gives rise to internal liquidity management in that firms structure compensation packages (Michelacci and Quadrini 2009) and accumulate cash (Bates et al. 2009) to avoid financial distress. At the same time, internal cash holdings exacerbate agency conflicts as they serve as a larger pool of cash from which insiders can divert (Jensen 1986). In this paper, we study how a firm optimally designs compensation and payout policies when it faces both financing and agency frictions.

To do so, we develop a dynamic, continuous time model in which a insider maximizes the value of a firm co-financed by dispersed outside investors (i.e., outside shareholders). We interpret the insider-outsider relationship broadly. For instance, the insider can be the manager or CEO of a firm financed by outside equity. Another interpretation is that the insider is the general partner within a private equity or venture capital fund. The insider could also represent the private equity or venture capital fund itself that affects firm performance, e.g., through its monitoring activity (Bloom et al. 2015, Kaplan and Stromberg 2001). We assume that this insider is risk-averse reflecting the notion that she maintains a concentrated exposure to the firm or faces private financial constraints. Importantly, the insider can lend to the firm, for example, by means of foregone wages, direct cash investment into the firm, or by pledging her firm ownership as loan collateral (Fabisik 2019), but this source of financing is both costly and limited due to the insider's risk-aversion. In contrast, the firm's dispersed outside investors are risk neutral but can only occasionally inject funds into the firm. As both insider's and outside investors' ability to inject funds into the firm is limited, the firm faces financial constraints. Financing constraints introduce the possibility of liquidation due to running out of cash, and so give rise to liquidity management.

The insider is subject to moral hazard and can divert cash for her own consumption (Jensen 1986).[1] The agency conflict constrains the ability to mitigate financing frictions by means of the insider's payout scheme. To deal with the moral hazard problem, the insider's payout scheme must comprise both short-term incentives through exposure to cash-flow shocks and long-term incentives through deferred payouts.

First, we find that risk-sharing and agency-frictions are closely intertwined. When a firm is relatively well capitalized, agency-frictions determine the performance-sensitivity of the insiders

[1] In contrast, the firm's dispersed outside investors do not directly affect firm performance (e.g., due to a free-rider problem).

stake, and we have performance-sensitivity pinned to the cash-diversion parameter as is common in the literature. In contrast, when a firm has low liquidity, risk-sharing considerations determine the performance-sensitivity. The contract stipulates a performance-sensitivity in excess what agency-frictions would require in order to lower the volatility of the firm's cash-holdings to either avoid liquidation (when the autarky value of the firms is negative) or costly temporary ownership of the firm by the insider (when the autarky value is positive).

Second, even considering only positive NPV firms,[2] agency conflicts keep some firms from attracting initial financing, while others are financed only under risk of subsequent liquidation. Interestingly, which firms can attract initial financing from outside investors does not depend on the severity of financial constraints and is soley determined by their feasibility in the second best case in which financially unconstrained outside investors need the insider to run the firm. Furthermore, which firms are at risk of liquidation is determined by the autarky value of running the firm to the insider. If she is willing to operate the firm in autarky, then the firm is never at risk of liquidation, as the insider provides financing under financial distress averting failure. Lastly, the expected time to failure is a monotonically increasing in how close the firm is to a positive autarky value.

Third, the optimal cash payout boundary, which is also the firm's starting cash balance, is non-monotone in cash-flow volatility, the NPV of the firm, and in the severity of the agency friction. When a firm is on the margin of attracting initial outside financing, for example due to high volatility, low NPV, or high agency friction, the payout boundary is low, implying that the resulting firm will be at imminent risk of failure. In this case, the insider is also required to contribute a significant fraction of the initial cash-balance. As a firm's feasibility increases,due to lower volatility, higher NPV, or lower agency friction, the firm's payout boundary increases, and its risk of failure decreases. Finally, beyond a certain point, the payout boundary starts decreasing. Even though the overall value of the firm increases monotonically, with lower volatility or higher cash-flow drift risk-management considerations are dominated by economizing on the carry-cost of cash. At the same time, the cash contribution of the insider shrinks, and beyond a point she is even able to extract cash initially.

Fourth, introducing infrequent access to frictionless capital markets, we find that such refinancing opportunities lead to endogenous flotation-cost due to agency frictions: for intermediate and higher levels of liquidity, the firm may not fully refinance to its payout boundary, in contrast to the

[2] Here, NPV is calculcated as the risk-neutral expectation, which is the first-best value of the firm when outsiders are unconstrained and insiders have no moral hazard, i.e., cash-flow drift over the discount rate.

the optimal policy in models of pure liquidity management when there is no costs to refinancing, as the insider's agency problem requires additional payoffs. For low levels of liquidity, refinancing to the payout boundary may indeed be optimal, but comes with cash bonus payments to the insider. The intuition is that promising bonus payments conditional on refinancing, via promise keeping, lowers required payouts outside of refinancing opportunities, thus preserving liquidity.

Notably, the presence of refinancing opportunities does not change the set of projects able to find financing, but *conditional* on financing increases the set of firms that are never at risk of failure as well as reduces the risk of failure for firms that are. The intuition is that improved access to outside financing also improves the insider's willingness to provide inside financing to the firm under financial distress, which ultimately mitigates failure risk.

Our work connects two related yet until now separate strands of the corporate finance literature. First, our paper adds to the literature on dynamic incentive problems. Recent contributions include DeMarzo and Sannikov (2006), Sannikov (2008), DeMarzo et al. (2012), Ai and Li (2015), Biais et al. (2010), Zhu (2012), Green and Taylor (2016), Varas (2017), He (2011), He et al. (2017), and Marinovic and Varas (2019b). The innovation of our paper is that we consider a financially constrained principal, i.e., the outside investors in our model.

Second, our model is linked to the theoretical literature on dynamic liquidity management within firms. Here, Bolton et al. (2011, 2013), Décamps et al. (2011); Decamps et al. (2016), Hugonnier et al. (2014), Gryglewicz (2011), Della Seta et al. (2019) and Hugonnier and Morellec (2017) are the closest references. Unlike our paper, these papers do not study dynamic compensation schemes and incentives. Our paper is also related to Abel and Panageas (2020) and Decamps et al. (2016). These papers demonstrate that an increase in persistent cash flow risk may increase or decrease target cash holdings. Unlike these papers, we find that the relationship between target cash holdings and risk is non-monotonic in a stationary cash flow environment if moral hazard is sufficiently severe.

Our paper also adds to the literature on investor activism. Important contributions include Admati et al. (1994), DeMarzo and Sannikov (2006), and more recently Marinovic and Varas (2019a).

In addition, important empirical and structural papers on corporate cash management and financial constraints include Whited and Wu (2006), Almeida and Campello (2007), Nikolov and Whited (2014), Lamont et al. (2001), Bates et al. (2009), Hennessy et al. (2007), and Nikolov et al. (2018). Closely related to our paper is Bolton et al. (2019) who analyze how to optimally structure compensation, investment and payout policies. However, their paper does not feature agency conflicts. Our paper also adds to the literature analyzing risk-sharing between firms and their workers, such as the theoretical studies of Berk et al. (2010) or Hartman-Glaser et al. (2019) or the empirical study of Guiso et al. (2005), who document that firms ensure their workers only partially against cash-flow risk.

1 Model Setup

Time t is continuous on $[0, \infty)$. We consider a firm that is owned by a risk-averse agent whom we call the insider. At time $t = 0$, the insider designs a payout agreement that includes a security she sells to competitive and risk-neutral outside investor (i.e., outsiders) and an inside stake within the firm that she retains. We interpret the insider-outsider relationship broadly. For instance, the insider can be the manager or CEO of a firm that raises funds by issuing outside equity or an entrepreneur who raises outside financing to start a firm. Another interpretation is that the insider is the general partner within a private equity (e.g., venture capital) fund. The insider could also represent the private equity (e.g., venture capital) fund itself that affects firm performance.

Preferences. The common discount rate is $r > 0$. The firm's outside investors are risk-neutral and (for the moment) cannot inject funds into the firm after time zero. That is, we assume that outside investors are diversified, have limited liability, and at the moment cannot buy additional equity after time zero. In contrast, the insider is risk-averse with CARA preferences in that her flow utility is given by

$$u(c_t) = -\frac{1}{\rho} \exp(-\rho c_t)$$

with risk aversion coefficient $\rho > 0$ and flow consumption c_t. Risk aversion reflects the notion that the type of insiders we consider in this paper typically have relatively undiversified exposure to the firm, that is, have concentrated ownership in the firm. Another interpretation is that the insider's risk-aversion broadly captures her private financial constraints.

Cash Flows and the Payout Agreement. The firm generates publicly observable and contractible cash flows dX_t each $[t, t + dt)$ with mean μ and volatility σ

$$dX_t = \mu dt + \sigma dZ_t, \tag{1}$$

where Z is a standard Brownian Motion. A payout agreement (i.e., contract) $\mathcal{C} = (Div, w)$ is defined by a cumulative stream of dividends Div_t promised to outside investors and a cumulative stream of transfers w_t promised to the insider, including dividends and transfers at the time of founding (i.e., from $t = 0^-$ to $t = 0$). In the baseline version of the model, we assume that it is not possible to raise external funds from new investors after the founding of the firm at time $t = 0$, for example, by issuing equity. That is, we assume $dDiv_t \geq 0$ for all $t \geq 0$, reflecting that outsider investors can only inject cash into the firm from $t = 0^-$ to $= 0$. We relax this assumption in Section 4 by introducing refinancing opportunities after time zero. Because the outside investors are unable to inject cash, all operating losses, dividends $dDiv_t$, and transfers to (and from) the insider dw_t must come out of the firm's internal cash balance, M_t.

Besides payouts, two other factors affect the accumulation of cash: First, the cash within the firm accrues interest at the rate $r - \delta$.[3] The wedge $\delta > 0$ represents so-called carry-cost of cash. The Appendix I.1 demonstrates how these costs can arise *endogeneously* as a result of agency conflicts.

Second, as in DeMarzo and Sannikov (2006), the insider can divert cash for her own private benefit, where db_t denotes the instantaneous cash diversion at time t. Thus, the dynamics of M_t are given by

$$dM_t = [(r - \delta)M_t + \mu]\, dt + \sigma dZ_t - dDiv_t - dw_t - db_t \tag{2}$$

Cash holdings cannot be negative, i.e., $M_t \geq 0$, $\forall t \geq 0$. This implies that in case M_t drops down to zero, the insider must either inject funds or liquidate the firm. Liquidation thus occurs at some stopping time $\tau \in [0, \infty]$, and $dDiv_t = dw_t = dX_t = 0$ for all $t > \tau$.

Moral Hazard. Cash-holdings M are publicly observable whereas cash-flow realizations X are privately observed by the insider. The insider can inefficiently divert cash db_t for her own use in which case she receives λdb_t dollars with $\lambda \in [0, 1)$. The insider's cash diversion db is not contractible and, if db_t is of order dt, then db_t is also not observable to outside investors.[4] At any time t, the insider can abscond with the entire cash balance (i.e., $db_t = -M_t$), triggering firm liquidation. That is, the insider's ability to commit is limited: the insider can only commit to payout agreements that deliver at least the same payoff as she would obtain after absconding with the firm's cash-balance. The limited commitment assumption rules out that the payout agreement

[3] The assumption is standard in the literature and is needed to preclude degenerate solutions, in which dividend payouts are indefinitely delayed.

[4] If db_t is of order dt and hence infinitesimal, then in fact db_t is not observable to outside investors. In contrast, if db_t is not infinitesimal, then outside investors can observe db_t.

may stipulate punishment to the insider after absconding with the firm's cash-balance.

The insider can maintain hidden outside savings (both positive or negative), denoted by S_t. Savings S accrue interest at rate r and are subject to changes induced by payouts dw_t, diverted cash db_t, and consumption c_t. This gives rise to the following

$$dS_t = rS_t dt + \lambda db_t + dw_t - c_t dt \tag{3}$$

Endowing the insider with the possibility to accumulate cash savings is needed to ensure consumption smoothing beyond any liquidation event. We normalize the insider's initial savings position to zero, i.e., $S_{0^-} = 0$, and assume that S_t adheres to standard regularity conditions, formally defined in Appendix A. For simplicity, the insider's savings are not subject to the carrying cost of cash $\delta > 0$. Indeed, the Appendix I.2 demonstrates that the analysis and the results remain largely unchanged if the insider's savings are subject to the interest rate $r - \delta$ rather than r.

In practice, the insider can inject cash into the firm, for example, by means of foregone wages, direct cash investment into the firm, or by pledging her firm ownership as loan collateral (Fabisik 2019), but this source of financing is costly and limited due to the insider's risk-aversion. Accordingly, cash contributions from the insider, $dw_t < 0$, are possible yet costly.

The Optimal Security Design Problem. Given a contract $\mathcal{C}$, the insider chooses consumption c and diversion policies b to maximize her lifetime utility. Let $U(\mathcal{C}; S_0)$ be the insiders indirect utility for a given contract $\mathcal{C}$ and a given level of (post-founding) savings S_0, i.e.,

$$U_0 = U(\mathcal{C}; S_0) = \max_{c_t, b_t} \mathbb{E}\left[\int_0^\infty e^{-rt} u(c_t) dt\right]. \tag{4}$$

Given the inefficiency of diversion, it is without loss of generality to restrict attention to contracts for which the insider finds it optimal to choose $db_t = 0$. We call such contracts incentive compatible.

Including initial contributions dw_{0^-} and $dDiv_{0^-}$, the optimal security design problem can then be compactly written as

$$\max_{\mathcal{C}} U(\mathcal{C}) \tag{5}$$

such that $\mathcal{C}$ is incentive compatible. At inception at time zero, the outside investors contribute in total $-dDiv_{0^-}$ dollars and the insider contributes $-dw_{0^-}$ dollars. As the insider's initial cash positions and the payout processes start at zero, i.e., $S_{0^-} = Div_{0^-} = w_{0^-} = 0$, the cash position

M_0 of the company after inception is given by

$$M_0 = -\left[dDiv_{0^-} + dw_{0^-}\right] = \underbrace{P_0}_{\text{Cash from outsiders}} + \underbrace{(-S_0)}_{\text{Cash from the insider}} \tag{6}$$

where the second equality follows from noting that concave utility results in smooth consumption by the insider so that $dw_0 = S_0 - S_{0^-} = S_0$. The first equality uses that because financial markets are competitive, the outside investors' initial cash contribution $-dDiv_{0^-}$ is equal to the competitive price of the security the insider issues, P_0.[5] That is,

$$P_0 = \mathbb{E}\left[\int_0^\tau e^{-rt} dDiv_t\right] = -dDiv_{0^-}. \tag{7}$$

Note that $-S_0 = -dw_0$ is the cash contribution (positive or negative) by the insider to endow the nascent firm with a cash balance of M_0: when $S_0 < 0$ the insider is contributing cash to the firm, whereas when $S_0 > 0$ the insider is extracting cash from the firm by transferring some of the sale proceeds into her savings account. However, *regardless* of the sign of S_0, the insider will always have skin in the game, i.e., a positive stake in the firm, due to promised future payouts.

2 Model Solution

In this section, we solve the model and derive the optimal payout agreement. First, we analyze the insider's problem and characterize conditions for the payout agreement to be incentive compatible. Next, we show how to collapse the state space of the problem. We then provide a heuristic argument that the security design problem we give in equation (5) is equivalent to one in which the insider maximizes the value of the claim she sells to outside investor. Finally, we characterize the dynamic optimization problem by an ordinary differential equation (ODE).

2.1 Incentive Compatibility and Promise Keeping

Define the insider's continuation value at time t by

$$U_t := \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)} u(c_s) ds\right]. \tag{8}$$

[5] The security sells at fair value, so that outsiders are not able to capture any of the NPV of the project. This can also be seen by rewriting $\mathbb{E}\left[\int_0^\tau e^{-rt} dDiv_t\right] = -dDiv_{0^-} \iff \mathbb{E}\left[\int_{0^-}^\tau e^{-rt} dDiv_t\right] = 0$.

To provide incentives, the insider's continuation value is sensitive to firm performance dX_t in that

$$dU_t = rU_t dt - u(c_t)dt + \beta_t(-\rho r U_t)(dX_t - \mu dt), \tag{9}$$

with exposure to cash-flow shocks (i.e., pay performance sensitivity) β_t.[6]

First, note that because the insider has access to a savings technology, she optimally smoothes her consumption, implying that her marginal utility is a martingale. As shown in the Appendix, the insider's first order condition with respect to consumption given the access to a savings account implies that $u'(c_t) = -\rho r U_t > 0$. Because in addition $u'(c_t) = -\rho u(c_t)$, it follows that $u(c_t) = rU_t$, so that by (9), the insider's continuation utility U_t is a martingale. Second, let us define the certainty equivalent W_t as the amount of wealth needed that would result in utility U_t if the insider only consumed interest on her wealth rW_t, that is,

$$W_t := \frac{-\ln(-\rho r U_t)}{\rho r}. \tag{10}$$

Note that W_t is the insider's continuation value in *monetary terms* while U_t is the insider's continuation value in *utility terms*. One can also derive that due to $u(c_t) = rU_t$, it follows that $c_t = rW_t$, in that the insider consumes the "interest" on her certainty equivalent.

Promise keeping constraint. Applying Ito's Lemma to W_t, we obtain

$$dW_t = \underbrace{\frac{\rho r}{2}(\beta_t \sigma)^2}_{\text{Risk-Premium}} dt + \beta_t(dX_t - \mu dt) \tag{11}$$

Because her payouts are sensitive to cash-flow shocks dX_t, the insider demands a risk premium, in that W_t has a drift equal to the required risk premium. Let $Y_t = W_t - S_t$ the insider's *deferred compensation*. Proposition 1 shows that:

$$Y_t = \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} \left(dw_s - \frac{\rho r}{2}(\beta_s \sigma)^2 ds \right) \right] \tag{12}$$

and

$$dY_t = rY_t dt + \frac{\rho r}{2}(\beta_t \sigma)^2 dt + \beta_t(dX_t - \mu dt) - dw_t. \tag{13}$$

[6]Formally, the existence of β_t is guaranteed by the Martingale Representation Theorem.

Indeed, Y_t is the present value of the insider's future transfers dw_s for $s \geq t$ adjusted for risk and thus can be interpreted as the insider's deferred compensation. We refer to Equation (13) as the *promise keeping constraint*. It means that current transfers dw_t must be accompanied by a commensurate change in future promised transfers dY_t (i.e., $dY_t/dw_t = -1$), in order to deliver the promised value at time t (i.e., Y_t) to the insider.

Flow incentive constraint. We can now characterize the incentive compatibility conditions that guarantee zero cash diversion. First, consider the insider's incentive to divert one dollar from the cash flow, that is, set $db_t = 1dt$. In this case, the insider gains λdt dollars but also reduces cash flow dX by $1dt$ dollars. By (11), her continuation value (in monetary terms) falls in response to the diversion by $\beta_t dt$, so that it is optimal not to divert from cash flow if and only if:

$$\beta_t \geq \lambda. \tag{14}$$

Likewise, the insider does not find it optimal to boost cash-flow (i.e., $db_t < 0$) as long as $\beta_t \leq 1$.

Stock incentive constraint. Next, consider the insider's incentive to divert a lumpy amount of cash from the firm. At time t, the payout agreement promises deferred compensation worth Y_t to the insider. Notably, since the insider can always abscond from the payout agreement, the maximal punishment she can receive is a loss in future payouts, worth Y_t. Specifically, if the insider absconds with the entire cash balance M_t, she gains λM_t and loses Y_t. Therefore, incentive compatibility requires:

$$Y_t \geq \lambda M_t. \tag{15}$$

Overall, (14) and (15) illustrate that, in order to deal with the moral hazard problem, the insider's payouts encapsulate both short-term incentives, by means of exposure cash flow shocks $dW/dX = \beta$, and long-term incentives by means of deferred payouts Y. Short-term incentives are costly, since the agent demands a risk premium for being exposed to cash flow shocks. Long-term incentives are costly because the insider requires a compensation for being paid later.

We summarize our findings in the following Proposition.

Proposition 1. *Under a incentive compatible payout agreement $\mathcal{C}$, the following holds:*

1. *U follows (9) and W follows (11)*

2. *Y follows* (13) *with integral representation* (12)

3. $\beta_t \in [\lambda, 1]$ *for all* $t \geq 0$.

4. $Y_t \geq \lambda M_t$ *for all* $t \geq 0$.

2.2 Benchmarks

To gain intuition, it is useful to estabish two benchmarks: (i) the complete absence of any outside investors, which we call *Autarky* or A, and (ii) outside investors without any financing frictions, which we call *Second Best* or SB. In both cases, the firm holds no cash as the carry-cost δ makes it inefficient to do so.

Autarky Benchmark. First, recall the integral representation of Y_t given in (12). For the case of *Autarky*, the insider must absorb all cash flow fluctuations, i.e., $\beta_t = 1$ and $dw_t = dX_t = \mu dt + \sigma dZ_t$. In the autarky benchmark, the firm (once initiated) is never liquidated and the insider's deferred compensation Y_t and the outside equity value P_t are both time-stationary, in that $Y_t = Y^A$ and $P_t = P^A$ for all $t \geq 0$. Plugging into (12) and evaluating the expectation under the assumption of no default yields

$$Y^{\mathrm{A}} = \frac{\mu}{r} - \frac{\rho\sigma^2}{2} \quad \text{and} \quad S_0^{\mathrm{A}} = P^{\mathrm{A}} = 0. \tag{16}$$

Also note that the insider will initiate (start) the firm in autarky only if the NPV of the project is in excess of her full risk-premium $\frac{\rho r(\sigma)^2}{2}$, that is, when $Y^A \geq 0$.[7]

Second Best Benchmark. Let us consider the case of the *Second Best*. Given that the outsiders have no financial constraints and are risk-neutral, it is optimal to expose the insider to minimal risk subject to incentive compatibility (14) as such exposure is costly, in that $\beta = \lambda$. In the second best benchmark, the firm (once initiated) is never liquidated and the insider's deferred compensation Y_t and the outside equity value P_t are both time-stationary, in that $Y_t = Y^{SB}$ and $P_t = P^{SB}$ for all $t \geq 0$. Further, the insider sells as much of the firm as she can, so $Y_0 = 0$ and $P_0 > 0$.[8] Setting (12) equal to zero, the insider's compensation dw must have a drift of $\mu_w = \frac{\rho r(\lambda\sigma)^2}{2}$ to compensate for her risk exposure. Thus, as the outsiders are the *direct* claimants to the residual cash-flows (recall there is no cash in the firm) and the firm never defaults (incentives are from local variation

[7] Our convention is that NPV refers to the valuation risk-neutral financially unconstrained investors.

[8] Note that $Y_0 = 0$ does not imply that the insider has no exposure to firm risk — she does as $\beta > 0$. It simply states that her expected value going forward from participating in the firm is 0.

in transfers dw and not from default), we have

$$Y^{\mathrm{SB}} = 0 \quad \text{and} \quad S_0^{\mathrm{SB}} = P^{\mathrm{SB}} = \frac{\mu}{r} - \frac{\rho\,(\lambda\sigma)^2}{2}. \tag{17}$$

Only firms with $P^{SB} \geq 0$ receive financing. It is clear that a firm that does not receive financing in the second best scenario without financial constraints, also does not receive financing in the baseline scenario with financial constraints. Therefore, to make the analysis interesting, we make the following assumption:

Assumption 1. *Parameters satisfy* $P^{SB} > 0$.

When there are no agency frictions and $\lambda = 0$, the insider does not absorb any risk and the second-best price P^{SB} is equal to the firm's net present value $NPV = \frac{\mu}{r}$.

2.3 The Optimal Payout Agreement

In the following sections, we analyze the full problem in the presence of outside investors and financial constraints. We derive an expression for the value of outside equity that depends on the three endogenous state variables of the problem: i) the insider's certainty equivalent W_t, ii) the insider's savings S_t, and iii) the firm's cash-holdings M_t,

2.3.1 Reduction of the State Space

We now provide a heuristic argument to reduce the state space of the problem. We verify this argument in the proof of Proposition 2. Given the insider's continuation value W_t, the insider's savings S_t, and the firm's cash-holdings M_t, the value of outside equity is given by a function $\hat{P}(M_t, W_t, S_t)$. Because of the insider's CARA preferences, there are no wealth effects. As such, W_t and S_t do not separately enter the problem and only their difference $Y_t = W_t - S_t$ matters. Thus, we are left with the two state variables (M_t, Y_t), and the outside equity value can be written in the form $\hat{P}(M_t, W_t, S_t) = \tilde{P}(M_t, Y_t)$.

Next, consider a state $(M_t, Y_t) = (M, Y)$ that lies in the endogenous state space, with positive cash $M > 0$. Observe that at any time t with $(M_t, Y_t) = (M, Y)$, the payout agreement can stipulate that the insider either receives payouts $M \geq dw_t > 0$ from the cash balance or injects cash into the firm, i.e., $dw_t < 0$. The insider is indifferent between all such transfers so long as they are accompanied by a compensating change in Y_t. More formally, in state $(M_t, Y_t) = (M, Y)$ with

$M > 0$, the promise keeping constraint (13) implies that the insider's total payoff at time t, which consists of current transfers dw_t and the present value of future transfers $Y_t + dY_t = Y_t - dw_t + o(dt)$, is independent of dw_t.

Because transfers dw change the cash balance M by dw, the insider is indifferent between remaining at a point (M, Y) in the state space and receiving a payout dw while moving to the point $(M - dw, Y - dw)$, as long as $M - dw \geq 0$. As transfers dw do not affect the insiders value, they cannot affect the outsiders' value function at the optimum either. Thus, $\tilde{P}(M, Y) = \tilde{P}(M - dw, Y - dw)$.

Change of variables. Next, we introduce the convenient change of variables

$$C = M - Y. \tag{18}$$

Thus, we can write $\tilde{P}(Y, M) = p(C, Y)$ for some function p. We can interpret C as the net cash position of the firm in that it is equal to the cash balance less the amount that accrues to the insider by means of promised payouts. Note that because payouts dw reduce both cash balance M and deferred payouts Y by the same amount dw, they do not affect the net cash position $C = M - Y$. Therefore, we are left with

$$p(C, Y) = \tilde{P}(Y, M) = \tilde{P}(Y - dw, M - dw) = p(C, Y - dw), \tag{19}$$

with $M \geq dw \geq 0$ and $M > 0$. As a result, $\frac{\partial}{\partial Y} p(C, Y) = 0$ implying that the outside equity value is independent of the current value of Y. This means that we can express the outside equality value $p(C, Y)$ as a function of C only so that $P(C) = p(C, Y) = \tilde{P}(Y, M)$ for all (M, Y) in the state space with $M > 0$. By continuity, it also follows that $P(C) = \tilde{P}(Y, M)$ for all (M, Y) in the state space.

2.3.2 The Optimal Security Design Problem Revisited

To translate the security design problem into a standard dynamic programming we problem, we argue that maximizing the insiders payoff is equivalent to maximizing the initial value of outside equity, $P(C_0)$. The insider maximizes her lifetime utility (4) or equivalently her lifetime payoff in monetary units $W_0 = W(\mathcal{C}, S_0)$. Hence the insider's problem at time zero reads:

$$\max_{\mathcal{C}} W(\mathcal{C}, S_0). \tag{20}$$

Recall that her savings at time zero are equal to the difference between the amount she raises from outside equity and the initial cash balance of the firm $S_0 = P_0 - M_0$. Now note that the insider's total payoff is equal to the sum of her savings and her deferred payouts, that is, $W_0 = S_0 + Y_0$, and the net cash position of the firm is equal to the difference between the firm's cash balance and the insiders deferred payouts, that is, $C_0 = M_0 - Y_0$. Using these relations, the insider's problem is then equivalent to

$$\max_{\mathcal{C}} P(C_0) - C_0. \tag{21}$$

To solve this problem, we first dynamically maximize the outside equity value $P(C_0)$ for a given level of initial net liquidity C_0 and then determine the optimal level of initial net liquidity, C_0.

2.3.3 The Hamilton-Jacobi-Bellman equation

We can now derive a Hamilton-Jacobi-Bellman (HJB) equation for the outside equity value and use it to solve for the optimal payout agreement and liquidity policy.

State dynamics. Using equations (2), (13), $M_t = C_t + Y_t$, noting that by optimality $db_t = 0$ and $c_t = rW_t$, we have

$$dC_t = \left[(r - \delta)C_t - \delta Y_t - \frac{\rho r}{2}(\beta_t \sigma)^2 + \mu\right] dt + (1 - \beta_t)\sigma dZ_t - dDiv_t. \tag{22}$$

Two features of the dynamics of net liquidity C_t are worth mentioning. First, note that we can rewrite the incentive constraint $Y \geq \lambda M = \lambda(C + Y)$ to $Y \geq \frac{C}{1-\lambda}$. While deferring payments to the insider through setting $Y > 0$ is necessary to maintain incentive compatibility, it requires the firm to hold more cash, which leads to an additional carrying-cost of cash $\delta Y dt$. Second, since cash flow shocks affect both cash holdings and the present value of the insider's payouts, the firm's net liquidity is less sensitive to cash flow shocks than cash, i.e., $\frac{dC_t}{dX_t} = 1 - \beta_t < 1 = \frac{dM_t}{dX_t}$.

Finally, note that it is always possible for the insider to decrease net cash C by stipulating positive dividend payouts $dDiv > 0$. In contrast, it is not possible to increase net cash C, as dividends $dDiv$ must be non-negative. Hence, C enters the insider's dynamic optimization problem as *state variable*. Because transfers to the insider dw are not sign-restricted, it is always possible for the insider to increase or decrease Y as long as cash-holdings M are positive. Thus, Y becomes a *control variable* for the insider's optimization. Therefore, the outside equity value is a function of C, $P(C)$, as already argued in section 2.3.1. Also note that the insider's transfers dw do not enter equation (22), but will be implicitly defined by the optimal choice of Y.

HJB equation. We derive the HJB equation for the firm's outside equity value $P(C)$ on the endogenous state space $[\underline{C}, \overline{C}]$ with $\overline{C} \geq \underline{C}$. We conjecture that dividend payouts only occur at some upper boundary $\overline{C}$, i.e., $dDiv = \max\{C - \overline{C}, 0\}$. In optimum, outside shareholders must earn expected returns $\frac{\mathbb{E}[dP(C)+dDiv]}{P(C)}$ equal to their discount rate rdt. On $(\underline{C}, \overline{C})$, there are no dividend payouts and an application of Ito's formula yields

$$rP(C) = \max_{\beta \geq \lambda, Y \geq \frac{\lambda C}{1-\lambda}} \left\{ P'(C) \left[(r-\delta)C - \delta Y - \frac{\rho r}{2}(\beta\sigma)^2 + \mu \right] + \frac{\sigma^2(1-\beta)^2}{2} P''(C) \right\}. \tag{23}$$

Dividend boundary $\overline{C}$. Note that it is always possible but not necessarily optimal to distribute dividends $dDiv \in (0, C)$ to outside shareholders, in that $P(C - dDiv) + dDiv \leq P(C)$. Taking $dDiv \to 0$, it follows that $P'(C) \geq 1$, meaning that the marginal value of liquidity always exceeds one and that the outside equity value increases in net liquidity. At the dividend boundary $\overline{C}$, the marginal value of liquidity is identical one. That is, the dividend boundary satisfies the following standard smooth pasting and super contact conditions

$$P'(\overline{C}) = 1 \quad \text{and} \quad P''(\overline{C}) = 0. \tag{24}$$

Because of $P'(C) \geq 1$, the outside equity value increases in C and, therefore, is low when the firm's liquidity C is low. That is, the firm undergoes financial distress when C is low and close to $\underline{C}$, whereas the firm is relatively unconstrained when C is high and close to $\overline{C}$.

As a further result, $P''(C) < 0$ for $C < \overline{C}$ in that the outside equity price reflects a risk premium. Although the outside investors are risk-neutral, they are effectively risk averse due to the possibility of financial distress. Dividends are therefore paid out exactly when the effective risk aversion and financial constraints vanish.

Lower boundary $\underline{C}$. Note that $C = 0$ implies $M = Y = 0$, in that at $C = 0$, the firm has run out of cash. Because the firm does not have access to outside financing through financial markets, at $C = 0$, the following two scenarios are possible: either the firm is i) kept alive with inside financing provided by the insider or ii) liquidated.

(i): The firm is *not* liquidated at $C = 0$. If $Y^A \geq 0$, the insider has positive private valuation $Y^A \geq 0$ for the firm and liquidating the firm at $C = 0$ is inefficient. Indeed, the insider can improve her payoff by financing the firm, until the firm's cash-holdings are positive again and the firm can be operated without financing from the insider again. The insider finances the firm in exchange for an increase of her deferred compensation Y and her firm ownership stake. As Section 5 discusses in more detail, there are instances with $Y > 0 = M$, $C = -Y$, and $0 > C \geq \underline{C}$. That is, the insider provides inside credit to the firm. For now, we assume that net cash does not become negative and impose that $\underline{C} = 0$. [9]

To avert liquidation at $C = \underline{C} = M = 0$, the insider covers the firm's losses and absorbs all cash-flow risk, hence i) the volatility of dC must be zero and ii) the drift of dC must be positive. These two requirements can be satisfied at $C = 0$ if and only if $Y^A = \frac{\mu}{r} - \frac{\rho\sigma^2}{2} \geq 0$. As a result, it is optimal not to liquidate the firm at $C = 0$ if and only if $Y^A \geq 0$. Because setting $\beta(0) = 1$ is optimal when $Y^A \geq 0$, we obtain the following boundary condition

$$\lim_{C \to 0} P''(C) = -\infty. \tag{25}$$

Even though there is no risk of firm liquidation, the firm accumulates cash to avoid costly financing from the insider. That is, the accumulation of cash allows to load-off risk from the insider by setting $\beta < 1$ and thus facilitates risk-sharing with outside investors.

(ii): When $Y^A < 0$, the firm is liquidated at $C = 0$, leading to the boundary condition

$$P(0) = 0 \tag{26}$$

with $\underline{C} = 0$. As liquidation entails deadweight losses and so is inefficient, the firm accumulates cash to avert liquidation.

Optimal controls (Y, β)**.** Next, from (23) we observe that it is costly to give the insider excess deferred payouts due to the cash cost of carry. When $C > 0$, then the incentive constraint (15) is tight and

$$Y = Y(C) = \frac{\lambda C}{1 - \lambda}. \tag{27}$$

[9] This assumption could capture that a firm's shareholders lack commitment to honor their obligations to creditors.

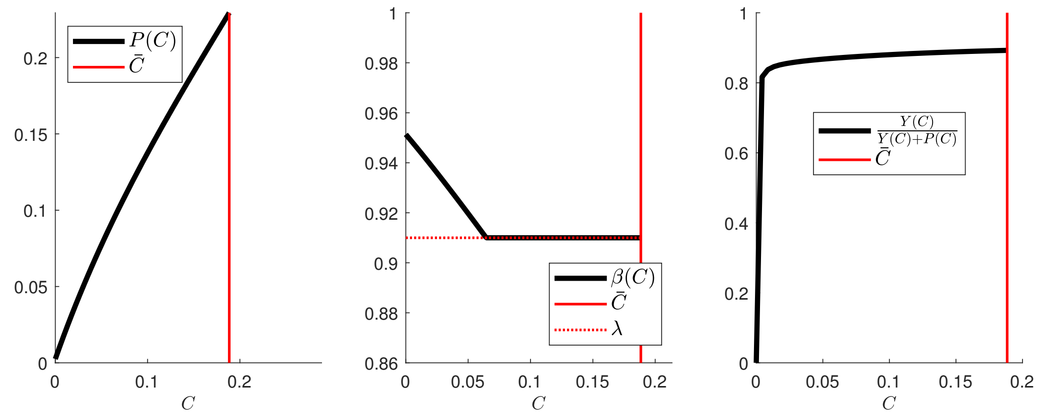

Figure 1: Numerical Example with exogenous lower bound $\underline{C} = 0$. The parameters are such that $Y^A > 0$, in that the firm is liquidated at $C = \underline{C} = 0$.

Using $M = C + Y$, we obtain that $M = M(C) = \frac{C}{1-\lambda}$. Maximizing with respect to insider's incentives β, the first-order conditions and the incentive compatibility constraint (14) imply that

$$\beta(C) = \max\{\lambda, \beta^*(C)\} \quad \text{where} \quad \beta^*(C) := \frac{-P''(C)}{\rho r P'(C) - P''(C)} < 1. \tag{28}$$

Increasing β not only provides incentives but also transfers risk to the insider and thus reduces the volatility of C, as can be seen in (22). This is beneficial because the outside equity value reflects a risk-premium.

We summarize our results in the following Proposition:

Proposition 2. *Let $\mathcal{C}$ solve (7) and stipulate that $\underline{C} = 0$. Then, the following holds true:*

1. *The outside equity value P solves (23) subject to $P'(\overline{C}) - 1 = P''(\overline{C}) = 0$*
2. *P is strictly concave $[0, \overline{C})$ with $P'''(C) > 0$.*
3. *At $C = 0$, the firm is either (i) is kept afloat by the insider if $Y^A \geq 0$, in which case $\lim_{C\to 0} P''(C) = -\infty$, or (ii) liquidated if $Y^A < 0$, in which case $P(0) = 0$.*
4. *The firm receives financing at time zero if and only $P^{SB} \geq 0$.*

The total firm value is given $P(C)+Y(C)$, which is the sum of the value of outside equity (i.e., the outside investor's stake) and the value of the insider's deferred compensation (i.e., the insider's stake). Figure 1 presents the numerical solution of the model, plotting the outside equity value $P(C)$, the sensitivity $\beta(C)$, and the insider's ownership stake $\frac{Y(C)}{Y(C)+P(C)}$ in the firm. We pick the parameters for the numerical solution following prior contributions in the literature (e.g., Bolton et al. 2011), as discussed in Appendix A.

Finally, note that when $P^{SB} \leq 0$ the firm is not financed, so that $\overline{C} = 0$. To see this, evaluate the HJB equation (23) at the dividend payout boundary $C = \overline{C}$, and using the fact that $P'(C) \geq 1$, we can show that the function $\overline{C} > 0$ if and only if $P^{SB} > 0$. That is, in the presence of financing constraints, a firm is able to attract financing at time zero if and only if it could also attract financing in the second best scenario *without* financing constraints.

As a result, firms with net present value $\frac{\mu}{r} < \frac{\rho(\lambda\sigma)^2}{2}$ are do not receive financing at all, firms with net present value $\frac{\mu}{r} \in \left(\frac{\rho(\lambda\sigma)^2}{2}, \frac{\rho\sigma^2}{2}\right)$ receive initial financing but face the risk of liquidation (i.e., failure) at a later point in time, and firms with net present value $\frac{\mu}{r} > \frac{\rho\sigma^2}{2}$ receive financing and are never liquidated as in addition to outside financing they receive inside financing under financial distress. Thus, whether a firm can attract financing at time zero depends on the severity of agency conflicts but not on the severity of financial constraints. While agency conflicts may prevent a profitable firm from being financed at time zero, they do not affect whether a firm — conditional on being financed at time zero — is subject to failure (i.e, liquidation) risk.

3 Analysis

3.1 Optimal Risk Sharing

In this section, we analyze how value is created through risk-sharing by investigating the insider's exposure β, also referred to as pay performance sensitivity. Recall that in the *Second Best*, there is no need to risk-share as the outside investors are unconstrained and risk-neutral, and thus optimally the insider is only exposed to the minimal amount of risk that maintains her incentives, i.e., $\beta = \lambda$.

To start with, consider a firm with no agency issues, i.e., $\lambda = 0$, but financially constrained outsiders. Optimal exposure is given by $\beta(C) = \beta^*(C)$. In this case, β is solely used for liquidity management, as exposing the insider reduces the risk of the reserves to $dC/dX = (1 - \beta)$. Constrained outside investors become increasing risk-averse ($P''(C) < 0$), so that as cash reserves dwindle, risk-sharing with the insider, even though expensive due to the insider's constant risk-aversion, becomes more prominent. Thus, $\beta^*(C)$ decreases in C (see Figure 1).

When both financial constraints and agency frictions are present, β will reflect both effects, (i) liquidity management through risk-sharing and (ii) incentive constraints. When $\beta(C) = \beta^*(C) > \lambda$, liquidity management considerations dominate, whereas when $\beta(C) = \lambda > \beta^*(C)$, incentive considerations dominate. As Figure 1 illustrates, liquidity management considerations dominate for low values of C, while incentive considerations dominate for higher values of C, so that $\beta(C)$ (strictly) decreases in C (for low C). At the same time, the insider's deferred compensation $Y(C) = \frac{\lambda C}{1-\lambda}$ and her ownership stake $\frac{Y(C)}{Y(C)+P(C)}$ both increase in net liquidity. In other words, the insider's long-term incentives captured by her firm ownership stake increases in C (good performance), while the insider's short-term incentives as captured by β decrease in C (after good performance), hence the insider's overall incentives become more long-term after good performance.

Last, payouts dw to the insider are implicitly defined as the residual that keeps $Y = \frac{\lambda C}{1-\lambda}$. Imposing $dY = \frac{\lambda}{1-\lambda} dC$, we have $dw = \mu_w dt + \sigma_w dZ + \xi_w dDiv$ with

$$\begin{aligned} \mu_w(C) &= \frac{1}{1-\lambda}\left[\frac{\rho r}{2}\left(\beta\left(C\right)\sigma\right)^2 + \frac{\lambda\delta C}{1-\lambda} - \lambda\mu\right] \\ \sigma_w(C) &= \left(\frac{\beta\left(C\right)-\lambda}{1-\lambda}\right)\sigma \\ \xi_w &= \frac{\lambda}{1-\lambda}. \end{aligned} \tag{29}$$

As a result, the insider covers losses $dZ < 0$ when $\sigma_w > 0$, which occurs when $\beta(C) > \lambda$. In other words, interim financing in response to cash-flow shocks occurs exactly when the risk-sharing considerations outweigh the agency issues. Thus, the insider is less likely to provide interim financing to the firm if agency issues are severe, yet more likely when the firm's liquidity (i.e., cash-holdings) is low.

We summarize our findings in the following corollary.

Corollary 1. *Let $\mathcal{C}$ solve (7). Then, the following holds true:*

1. *There exists $C' \in [0, \overline{C})$ with $\frac{\partial \beta^*(C)}{\partial C} < 0$ on $[C', \overline{C}]$. If σ is sufficiently small, then $C' = 0$.*

2. *There exists unique $\hat{C} \in [0, \overline{C}]$ with $\beta(C) > \lambda$ for $C < \hat{C}$. If λ is sufficiently small, then $\hat{C} > 0$.*

3. *Transfers $dw = \mu_w(C)dt + \sigma_w(C)dZ + \xi_w dDiv$ are characterized by (29).*

3.2 Cash Holdings, Agency Conflicts, and Risk

In this section, we discuss how cash flow risk and agency conflicts affect the target cash holdings of the firm. To start with, recall that when i) $Y^A \leq 0$, the firm holds cash to avert inefficient liquidation, and ii) when $Y^A > 0$, the firm holds cash to facilitate risk-sharing with outside investors,

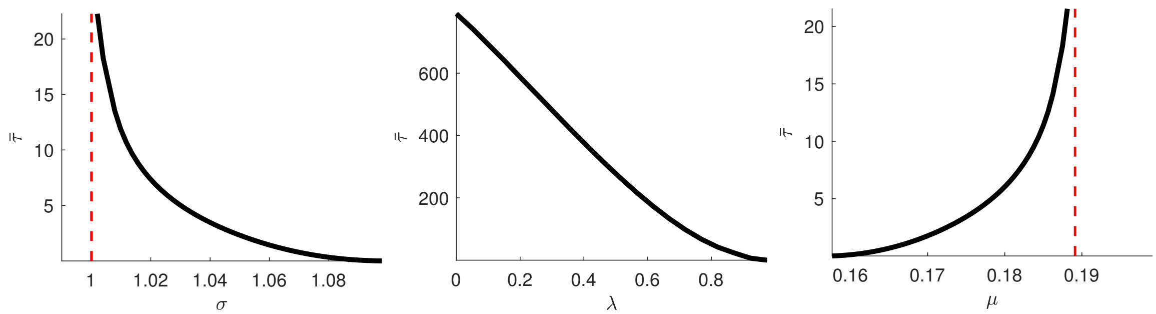

Figure 2: Comparative Statics of the expected time to failure, $\overline{\tau} := \mathbb{E}_0\tau$. In this graph, we consider that $Y^A < 0$, so $\tau < \infty$. On the other hand, when $Y^A \geq 0$, then $\tau = \infty$. The dashed red line denotes the locus $Y^A = 0$.

leading to positive target cash-holdings $\overline{M}$. The target cash-holdings $\overline{M} = \frac{\overline{C}}{1-\lambda}$ are plotted in the upper row of Figure 3. The target cash-holdings are proportional to $\overline{C}$, so the comparative statics coincides with the comparative statics of $\overline{C}$ as long as we do not vary λ. The firm's target cash-holdings $\overline{M}$ proxy a firm's average cash-holdings, since the firm spends most of its lifetime in states C (or M) close to $\overline{C}$ (or $\overline{M}$), as shown in Bolton et al. (2013).

In our model, the payout boundary $\overline{M}$ (or $\overline{C}$) is non-monotonic in cash flow volatility σ. The intuition for this result is that an increase cash flow risk has two effects in our model. First, as is standard, increasing σ increases the risk the firm will exhaust it's current stock of liquidity, raising the risk of liquidation or constraining efficient risk-sharing. Close to the payout boundary, this effect increases the marginal value of cash, and thus delays payout. The new effect in our model is that an increase in cash-flow risk σ increases the cost of providing incentives to the insider, reducing the value of operating the firm. Due to moral hazard $\lambda > 0$, this holds true even when there are no financial constraints, in that $\frac{\partial P^{SB}}{\partial \sigma} > 0$ if and only if $\lambda > 0$. Hence, in response to an increase in cash-flow volatility σ, i) firm liquidation becomes less inefficient and ii) the gains from risk-sharing with outside investors decrease. Either way, the benefits of hoarding cash to facilitate risk-sharing with outside investors or averting liquidation decrease in σ, reducing the target level $\overline{M}$ (and $\overline{C}$).

When the agency problem is mild, that is, when the efficiency of cash diversion λ or the volatility of cash flows is small, the liquidity effect dominates the agency cost effect and an increase in cash flow volatility increases the payout boundary. When the agency problem is severe, that is, when both the efficiency of cash diversion λ and the volatility of cash flows are large, the agency cost effect dominates and an increase in cash flow volatility decreases the payout boundary. We also show in Figure 2 that consistent with the above intuition, an increase in cash-flow risk σ increases the risk of liquidation by increasing the expected time to liquidation $\overline{\tau} = \mathbb{E}\tau$ when $Y^A < 0$.

In addition, the payout boundary $\overline{C}$ (and the expected time to liquidation $\overline{\tau}$) decreases in the extent of moral hazard λ (when $Y^A < 0$). In contrast, the relationship between optimal cash holdings $\overline{M} = \frac{\overline{C}}{1-\lambda}$ and the agency parameter λ is ambiguous, in that $\overline{M}$ increases in λ for low values of λ and decreases for larger values of λ. The reason is that moral hazard affects target cash-holdings in two ways. First, cash within the firm exacerbates moral hazard, reducing the return to holding cash and therefore the optimal target cash-level when $\lambda > 0$. Second, cash within the firm is needed to be able to incentivize the insider by means of deferred compensation while maintaining sufficient financial slack, increasing the optimal target cash-level when $\lambda > 0$. More severe moral hazard requires more deferred compensation Y, thereby increasing expected future payouts to the insider. For a given level of cash M, this reduces the net cash position C of the firm. That is, the provision of incentives by means of deferred compensation requires additional cash holdings that collateralize the insider's deferred compensation (i.e., stake) but do not improve the firm's liquidity position. For low levels of λ the second effect (i.e., the agency effect) dominates, while for high levels of λ the second effect (i.e., the liquidity effect) dominates, thus $\overline{M}$ follows an inverted U shaped pattern in λ.

Last, the payout boundary $\overline{M}$ is non-monotonic in μ, while the expected time to default $\overline{\tau}$ increases in μ. An increase in μ raises i) the efficiency losses associated with liquidation and ii) the gains of risk-sharing with outside investors, improving incentives to accumulate cash. On the other hand, an increase in μ also boosts average earnings and thus the drift of dC, thereby making precautionary cash-holdings less vital. We formalize these results in the following Corollary.

Corollary 2. *For the net-cash payout boundary $\overline{C}$ and the cash payout boundary $\overline{M}$:*

1. $\frac{d\overline{C}}{d\sigma}, \frac{d\overline{M}}{d\sigma} > 0$ *for σ or λ sufficiently small.* $\frac{d\overline{C}}{d\sigma}, \frac{d\overline{M}}{d\sigma} < 0$ *for σ and λ sufficiently large.*

2. $\frac{d\overline{C}}{d\lambda} < 0$.

3. $\frac{d\overline{M}}{d\lambda} > 0$ *for ρ or λ sufficiently small.* $\frac{d\overline{M}}{d\lambda} < 0$ *for ρ and λ sufficiently large.*

3.3 Initial Investments and Value Creation

Next, we will analyze in more detail how the presence of outside investors creates value beyond allowing projects with net present value $\frac{\mu}{r} \in \left(\frac{\rho}{2}(\lambda\sigma)^2, \frac{\rho}{2}\sigma^2\right)$ to be initiated.

Initial net cash position. First, to maximize the outside equity value, the optimal initial net cash C_0^* position is found via (21). With $P(C)$ defined by the solution to (23) and its boundary conditions, we can now maximize w.r.t. C_0. Absent any constraints, the first-order condition yields the optimal initial net cash position

$$C_0^* = \arg\max_{C_0} P(C_0) - C_0 = \overline{C} \tag{30}$$

where we used $P'(C) > 1, \forall C \in \left[0, \overline{C}\right)$ and the smooth pasting condition (24). Thus, the firm is optimally endowed with an amount of net cash that positions it at the dividend payout boundary.

Initial cash contribution. At time zero, the insider sells a security worth $P(C_0) = P(\overline{C})$ to the outside investors. Even though the insider always has a positive initial continuation value $Y_0 > 0$, she might or might not contribute cash initially. Her initial contribution is simply the negative of her savings, $-S_0$. For any given initial position C_0, we have

$$S_0 = P(C_0) - M(C_0) = P\left(\overline{C}\right) - \frac{\overline{C}}{1-\lambda}. \tag{31}$$

That is, when $S_0 < 0$ ($S_0 > 0$), the insider contributes/injects cash (takes cash out) at time zero. We also analyze the fraction $\frac{-S_0}{M_0}$ of initial cash M_0 that is contributed by the insider (rather than by outside shareholders). In both the autarky and second best benchmark, there is no role for cash-holdings. In the autarky benchmark, the insider retains the firm, in that $S_0^A = 0$, while in the second-best benchmark the insider sells the firm for the price P^{SB}, in that $S_0^{SB} = P^{SB}$.

Next, let us investigate how much value is created by starting the firm. To this end, recall that the value of the firm is given by $P(C)+Y(C)$, i.e., by the sum of outside equity value $P(C)$ and the insider's stake $Y(C)$. Then, we can express initial value created, APV, as the difference between firm-value after founding and the initial cash outlay,

$$\text{APV} = \underbrace{P(C_0) + Y(C_0)}_{\text{Firm value at } t=0} - \underbrace{(P(C_0) - S_0)}_{\text{Cash contribution at } t=0} = Y(C_0) + S_0 = P(C_0) - C_0. \tag{32}$$

In the second best, we have $Y_0^{\text{SB}} = \overline{C}^{\text{SB}} = 0$ so that $\text{APV}^{\text{SB}} = S_0^{\text{SB}} = P^{\text{SB}}$, whereas in autarky, we have $P^{\text{A}} = \overline{C}^{\text{A}} = 0$, so that $\text{APV}^{\text{A}} = Y^{\text{A}}$. Naturally, we then must have $\text{APV}^{\text{SB}} > \text{APV} > \text{APV}^{\text{A}}$ in any other scenario.

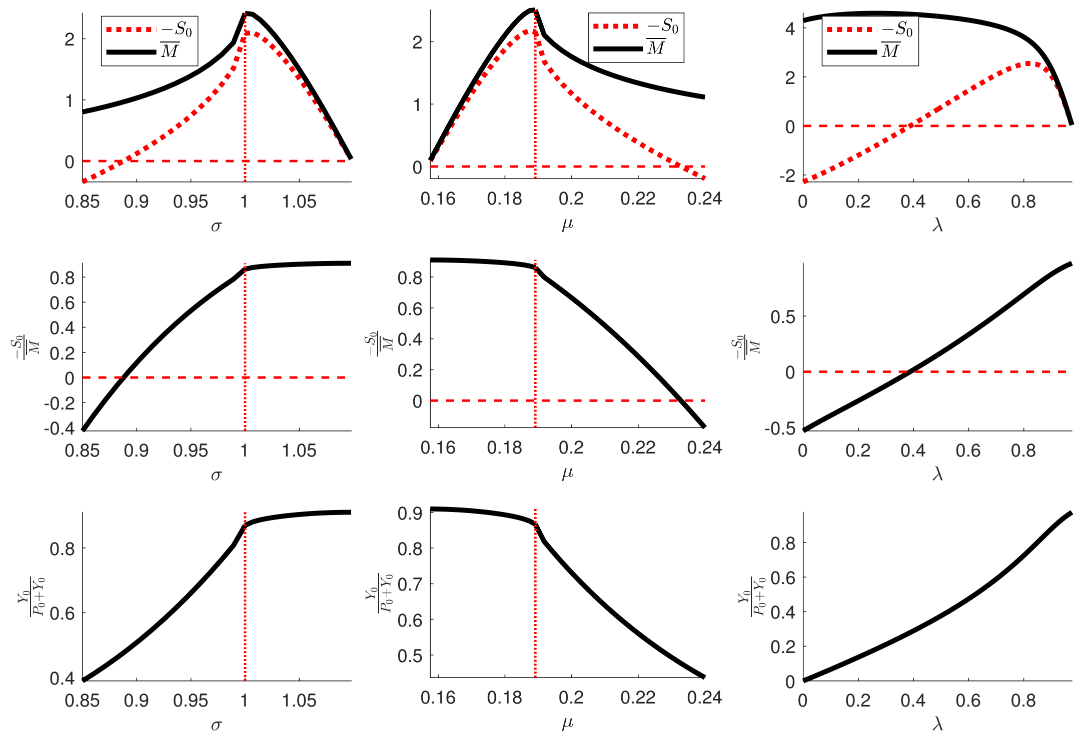

Figure 3: Comparative Statics Initial Contribution S_0. The choice of the baseline parameters follows prior contributions in the literature and is discussed in Appendix A. The dotted red-line depicts the locus $Y^A = 0$.

Initial stake. Finally, let us consider the insider's initial ownership stake of the insider in total firm value, $\frac{Y(C_0)}{P(C_0)+Y(C_0)}$, or in other words, how much of firm value $P(C_0) + Y(C_0)$ the insider retains. There is a subtle distinction between this retention stake and continued ownership in the firm. In fact, the insider's stake evolves dynamically: even if the insider's ownership retention equals zero at some point in time (i.e., $Y_t = 0$), she might ownership of the firm at a later point in time. The next section discusses the insider's dynamic ownership and its relation to agency problems more in detail.

Figure 3 illustrates the comparative statics with respect to σ, μ, and λ. The top row of Figure 3 shows the comparative statics of $-S_0$ (i.e., the insider's initial contribution) with respect to (σ, λ, μ). For σ and λ (μ), the right (left) edge of the graph is the point at which the project is only marginally able to attract outside investors, that is, where $P^{SB} \geq 0$ is just binding. We see that $-S_0$ converges to 0 from above as we approach the no-funding point at which $P^{SB} = 0$. That is, projects with low value to outside investors require the insider to contribute cash. Projects that are very valuable, i.e., that have a high P^{SB}, allow the investor to extract cash at founding. Overall, the insider is more likely to contribute cash (take cash out) at time zero, when the magnitude of agency conflicts λ is high (low) or the cash-flow volatility σ is high (low). The middle row of Figure 3 depicts that the fraction of cash the insider contributes (i.e., $-S_0/M_0$) is high when i) agency conflicts (λ) are severe, cash-flow volatility (σ) is high, or when the firm's profitability (μ) is low. Because the insider is compensated for her cash contributions by a larger initial ownership stake in the firm, the insider's initial ownership stake in the firm $\frac{Y_0}{Y_0+P_0}$ is high when i) agency conflicts (λ) are severe, cash-flow volatility (σ) is high, or when the firm's profitability (μ) is low (see bottom row of Figure 3).

4 Refinancing Opportunities

In our baseline model, the insider can only raise financing from outside investors at the outset of the model, i.e., at $t = 0$. In this section, we relax this assumption and analyse the impact that the possibility of refinancing has on the optimal payout agreement. To allow for refinancing in a tractable manner, we assume that opportunities to sell claims to new outside investors are infrequent and random as in Hugonnier et al. (2014). Over a short interval of time $[t, t + dt]$, the firm finds new outside investors with probability πdt, i.e., refinancing opportunities arrive according to the Poisson process Π_t with intensity $\pi \geq 0$. Note that as $\pi \to \infty$, we are essentially in the second best scenario. At refinancing times, the firm has unrestricted access to capital markets and there are no further cost to refinancing — the firm can issue equity at a fair price to raise cash. Further, we assume that the firm's existing owners appropriate all the surplus that refinancing generates.[10]

In particular, when the firm raises an amount Δ^M of cash from new outside investors, these outside investors obtain equity worth exactly Δ^M. In general, the payout agreement stipulates a reward to the insider upon successful refinancing, $d\Pi_t = 1$. This reward consists of an increase in deferred payouts, $\gamma_t = dY_t/d\Pi_t > 0$ with $Y_{t+dt} = Y_t + \gamma_t$ (if $d\Pi_t = 1$), and a cash bonus α_t paid out immediately. After paying α_t the firm is left with newly raised funds $\Delta_t := \Delta_t^M - \alpha_t$, so $M_{t+dt} = M_t + \Delta_t$ (if $d\Pi_t = 1$). Note that while refinancing alleviates the firm's financial constraints, it also increases the cash balance from which the insider can divert and thus exacerbates moral hazard. As a result, refinancing is subject to moral hazard and the rewards α and γ are set to provide sufficient incentives to the insider.

We derive the incentive compatibility that applies during the refinancing event. As a first, step, note that in general, the payout agreement could stipulate that the insider contribute funds during refinancing, leading to a lumpy cash contribution by the insider (i.e, $dw = -\alpha < 0$). The following Lemma shows that due to the insider's moral hazard problem, $\alpha \geq 0$ is optimal.

Lemma 1. *In optimum:* $\alpha \geq 0$.

The intuition behind this Lemma is that the insider may due to her moral hazard problem always renege to contribute funds during refinancing, so the stipulation of co-investment by the insider (i.e., $\alpha < 0$) cannot resolve the moral hazard problem during refinancing.

The moral hazard problem is now as follows. When a refinancing opportunity arises over $[t, t + dt]$, that is, $d\Pi_t = 1$, the insider may abscond with the total cash-balance $M_t + \Delta_t^M$, just after outside investors have contributed funds. Absconding yields i) a bonus payment, if $\alpha_t > 0$, and ii) diverted funds $\lambda M_{t+dt} = \lambda(M_t + \Delta_t)$, whereas complying to the payout agreement yields payoff $Y_{t+dt} = Y_t + \gamma_t + \alpha_t$. This leads to the incentive condition

$$\lambda(M_t + \Delta_t) \leq Y_t + \gamma_t. \tag{33}$$

We see that α does not affect the incentive constraint (33), but it can nevertheless play a role in

[10] If new investors and existing investors were to split the surplus according to the Nash-Bargaining protocol with respective weights $\eta, 1 - \eta$, then the problem is equivalent to one where the arrival rate is altered from π to $\eta\pi$, so that the choice $\eta = 1$ is indeed without loss of generality. See Hugonnier et al. (2014) for a more formal argument.

liquidity management as we will discuss below. Hence, the appropriate incentives during refinancing require promising the insider a sufficiently large deferred compensation package $Y_t + \gamma_t$ just after refinancing, implying that Y_t or γ_t must be large. Since the incentive constraint (33) tightens when more funds Δ_t are raised, the firm might decide to raise less funds due to agency conflicts. Thus, at the optimum, (33) holds with equality and $\alpha_t \geq 0$, so as to minimize the agency costs associated with the payout agreement. In addition, we consider in the following that $Y = \frac{\lambda C}{1-\lambda}$ is optimal, in that (15) is tight.

The insider's payouts can now be expressed as

$$dW = \frac{\rho r}{2}(\beta_t \sigma)^2 dt + \beta_t \sigma dZ - \frac{\pi\left[1 - e^{-\rho r(\alpha_t + \gamma_t)}\right]}{\rho r} dt + (\alpha_t + \gamma_t) d\Pi. \tag{34}$$

Because the insider shareholders is paid $\alpha_t + \gamma_t \geq 0$ upon refinancing, by promise keeping she requires lower flow transfers, i.e., W_t features a lower required drift by $\frac{\pi\left[1-e^{-\rho r(\alpha_t+\gamma_t)}\right]}{\rho r} > 0$. Essentially, $\alpha_t + \gamma_t > 0$ shift some of the insider's payouts from distress states towards states in which the firm is flush with liquidity, but since the insider has CARA utility the drift adjustment is non-linear in $\alpha_t + \gamma_t$. The dynamics of C then follow

$$dC = \mu_C dt + (1 - \beta_t) dZ + (\Delta_t - \gamma_t) d\Pi - dDiv \tag{35}$$

with

$$\mu_C := (r - \delta)C - \delta Y + \mu - \frac{\rho r}{2}(\beta_t \sigma)^2 + \frac{\pi\left[1 - e^{-\rho r(\alpha_t + \gamma_t)}\right]}{\rho r}. \tag{36}$$

By standard arguments, the HJB equation becomes:

$$rP(C) = \max_{\beta \geq \lambda, Y \geq \frac{\lambda C}{1-\lambda}, \alpha, \gamma, \Delta} \left\{ P'(C)\mu_C + \pi\left[P(C + \Delta - \gamma) - P(C) - \Delta - \alpha\right] + P''(C)\frac{\sigma^2(1-\beta)^2}{2} \right\} \tag{37}$$

subject to (33) and the standard boundary conditions $P'(\overline{C}) - 1 = P''(\overline{C}) = 0$ with payout boundary $\overline{C}$, i.e., $dDiv = \max\{C - \overline{C}, 0\}$. As refinancing opportunities are infrequent, financing frictions (continue to) prevail so that the value function reflects a risk premium in that $P''(C) \leq 0$. We characterize the optimal controls, denoted by $\alpha(C)$, $\beta(C)$, and $\gamma(C)$, below.

Finally, we consider the lower boundary $\underline{C}$, where we stipulate that $\underline{C} = 0$. The boundary conditions at $\underline{C} = 0$ depend on whether the firm is liquidated or not and are as follows:

1. The firm is *not* liquidated at $C = 0$. To avert liquidation, the following two conditions must

be satisfied at $C = 0$: i) the volatility of dC vanishes and ii) the drift of dC is positive. To satisfy the first condition, it must hold that $\beta(0) = 1$. Evaluating the drift of C at $C = 0$ under the assumption that $\beta = 1$, the second condition requires

$$\mu_C(0) = \mu - \frac{\rho r}{2}\sigma^2 + \frac{\pi\left[1 - e^{-\rho r[\alpha(0)+\gamma(0)]}\right]}{\rho r} \geq 0. \tag{38}$$

As setting $\beta(0) = 1$ must be optimal, it follows that $\lim_{C\to 0} P''(C) = \infty$.

2. The firm is liquidated at $C = 0$. If (38) is violated, the firm cannot avoid liquidation at $C = 0$, and the boundary condition $P(0) = 0$ applies.

Proposition 3. *Stipulate that $\underline{C} = 0$. Then, the following holds:*

1. *W evolves according to (34) and C evolves according to (35) for processes α, γ, Δ*

2. *The outside equity value solves (37) on $[0, \overline{C}]$ subject to $P'(\overline{C}) - 1 = P''(\overline{C}) = 0$. For $C = 0$, (26) holds, if $\tau < \infty$ is optimal, and (25) holds otherwise. Dividends are given by $dDiv = \max\{C - \overline{C}, 0\}$*

3. *P is strictly increasing and concave on $[0, \overline{C})$*

For convenience, let us define the dynamic target refinancing level

$$C^*(C) := C + \Delta(C) - \gamma(C). \tag{39}$$

Due to $Y = \frac{\lambda C}{1-\lambda}$, (33) reduces to:

$$\gamma(C) = \lambda\Delta(C) = \frac{\lambda\left[C^*(C) - C\right]}{1-\lambda}. \tag{40}$$

As a result, the optimization of the firm value becomes

$$rP(C) = \max_{\beta\geq\lambda,\alpha,C^*}\left\{P'(C)\mu_C + \pi\left[P(C^*) - P(C) - \frac{C^* - C}{1-\lambda} - \alpha\right] + P''(C)\frac{\sigma^2(1-\beta)^2}{2}\right\}, \tag{41}$$

with $Y = \frac{\lambda C}{1-\lambda}$

4.1 Moral Hazard and Cash Bonuses

We now examine how the agency problem affects refinancing decisions. On one hand, as the insider requires a reward by means of deferred payouts for incentive purposes, refinancing now

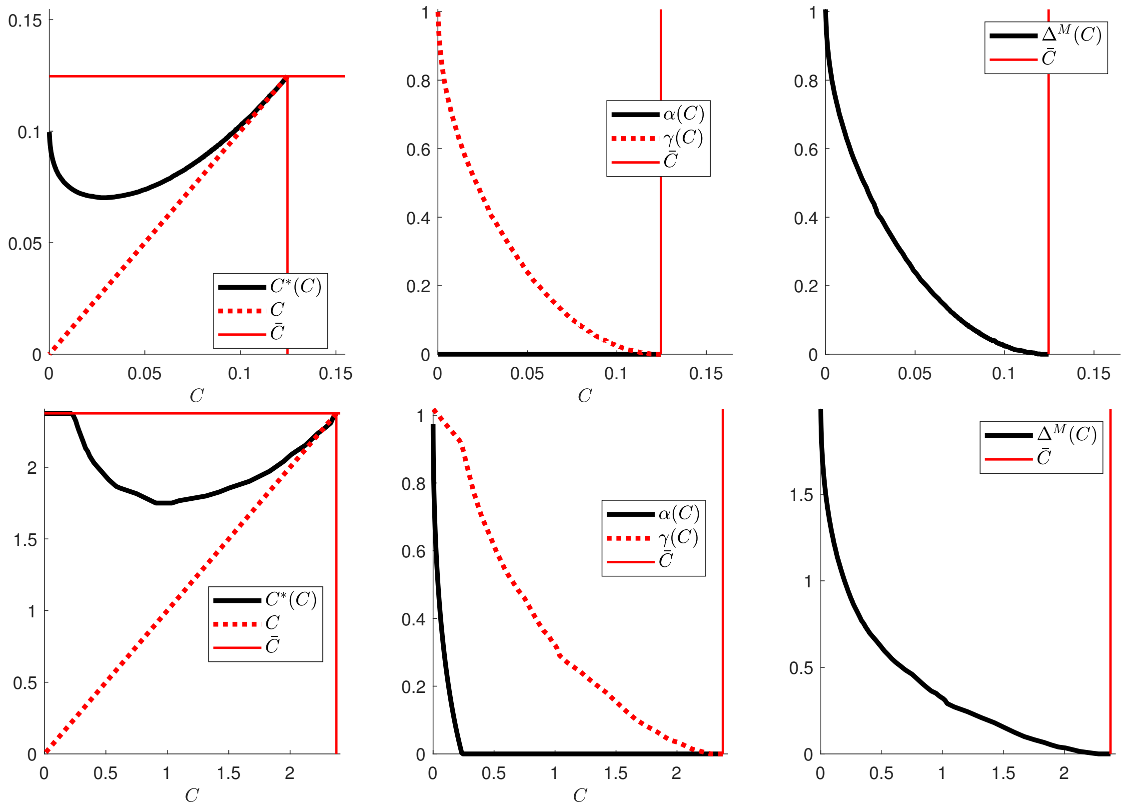

Figure 4: Numerical Example with refinancing opportunities, $\pi = 0.1$. The choice of the baseline parameters follows prior contributions in the literature and is discussed in Appendix A. The upper panel uses $\lambda = 0.91$. The lower panel uses $\lambda = 0.35$.

entails agency costs. On the other hand, rewarding the insider during a refinancing event reduces her required payouts before and so relaxes financing constraints. As a result, two scenarios are possible: First, a scenario in which agency considerations dominate liquidity management ones, and $\gamma > 0 = \alpha$. Second, a scenario in which liquidity management considerations dominate agency ones, and $\alpha, \gamma > 0$. This is analogous to the risk-sharing versus incentive trade-off that surfaced when we discussed the optimal β, but unlike there, the refinancing effect will operate through trading off the drift versus jump loadings instead of the drift versus volatility loadings.

Agency dominates. Let us first consider the case where agency considerations dominate, so that $\alpha = 0$. Then, the insider's reward is given by

$$\gamma(C) = \lambda \Delta = \frac{\lambda[C^*(C) - C]}{1 - \lambda} \quad \text{and} \quad \alpha = 0. \tag{42}$$

The candidate $C^*(C) \leq \overline{C}$ solves the following first-order-condition:

$$P'(C^*(C)) = 1 + \underbrace{\frac{\lambda}{1-\lambda}\left(1 - P'(C)e^{-\rho r \frac{\lambda}{1-\lambda}[C^*(C)-C]}\right)}_{\text{Flotation Costs}} \tag{43}$$

The candidate function $C^*(C)$ is indeed the equilibrium function if and only if

$$\overline{C} - C > \frac{\ln(P'(C))(1-\lambda)}{\rho r \lambda} \tag{44}$$

That is, only if (44) holds do we have have $C^*(C) < \overline{C}$ and consistency with $\alpha = 0$. Note that the flotation costs of refinancing are state dependent and dynamically evolve over time. Even though agency considerations dominate, stipulating $\gamma(C) = \frac{\lambda(C^*(C)-C)}{1-\lambda} > 0$ decreases required flow payouts to the insider and so induces risk-sharing benefits, increasing the drift of dC and thus lowering the (effective) agency costs of refinancing.

Liquidity dominates. Suppose instead that condition (44) is violated. Then, it must be that the candidate solution $C^*(C) > \overline{C}$ in (43). But refinancing to any point above $\overline{C}$ would lead to immediate dividend as well as transfer payouts to bring C down to $\overline{C}$. In the following, we thus "net out" the dividend payments[11] to re-write this as a bonus cash payment $\alpha > 0$, an increase in continuation value $\gamma > 0$, and a a target refinancing level $C^*(C) = \overline{C}$. Thus, liquidity management concerns outweigh agency issues, leading to $C^*(C) = \overline{C}$,

$$\gamma(C) = \frac{\lambda(\overline{C}-C)}{1-\lambda} \quad \text{and} \quad \alpha(C) = \frac{\ln(P'(C))}{\rho r} - \frac{\lambda(\overline{C}-C)}{1-\lambda}. \tag{45}$$

The intuition for why bonuses are used in addition to promised continuation value is as follows: when the firm has low liquidity C, outside investors are very risk averse as $P''(C) < 0$. Promising payments to the agent beyond the IC constraint (33) in the form of $\alpha > 0$ does not affect the IC constraint but increases the drift of C. Even though increasing the drift is *costly* — due the insider's risk-aversion, μ_C increases less than one-for-one with the expected bonus payments $\pi\alpha$ — when the firm has very low liquidity this may be beneficial. In essence, by stipulating $\alpha > 0$, payouts to the insider are delayed until the firm is again flush with liquidity.

Figure 4 numerically illustrates the optimal refinancing policies, both under severe agency conflicts ($\lambda = 0.91$; upper three panels) and mild agency conflicts ($\lambda = 0.3$; lower three panels). In

[11] The dividend payments would simply be the outside investors paying dividends to themselves for no value gain.

both cases, liquidity management and agency considerations shape the refinancing policies. In the upper three panels of Figure 4, severe moral hazard leads to $C^*(C) < \overline{C}$ and $\alpha(C)$ for all C, so that agency considerations dominate liquidity management ones. However, the refinancing target $C^*(C)$ follows a U-shaped pattern in C and, in particular, is high under financial distress (i.e., for low C), reflecting that the agency costs of refinancing are low under financial distress.

Under relatively mild agency conflicts (see lower three panels of Figure 4), liquidity management considerations dominate agency ones when C is low, whereas agency considerations dominate liquidity management ones when C is high. As a result, the refinancing target $C^*(C)$ is non-monotonic in C and the bonus payout $\alpha(C)$ decreases in C down to zero. That is, for low C, the refinancing target equals $C^*(C) = \overline{C}$ and the insider receives a bonus payout $\alpha(C) > 0$ upon successful refinancing. In contrast, for high C, the refinancing target is low with $C^*(C) < \overline{C}$ and there is no bonus payout. Regardless of that financially constrained firms tend to raise more cash upon refinancing as $C^*(C) - C$ is monotone in C, as can be seen for example in the left panel of Figure 4.[12]

4.2 Firm Formation, Liquidation, and Refinancing

Let us now investigate what firms can attract initial funding in a world with refinancing opportunities. First, the set of firms that receive financing in the first place is independent of π, in that a firm receives financing if and only if $P^{SB} \geq 0$.[13] That is, the bound to attract funding is the same as in the *Baseline*, which was already identical to the one in the *Second Best*, (17), and clearly as $\pi \to \infty$ our refinancing case converges to the *Second Best*.

Second, a firm is never liquidated if and only if (38), that is, when

$$\mu_C(0) = \mu - \frac{\rho r}{2}\sigma^2 + \frac{\pi\left[1 - e^{-\rho r[\alpha(0)+\gamma(0)]}\right]}{\rho r} \geq 0$$

holds. Note that the term in square brackets on the RHS of the second inequality is positive. Thus, refinancing opportunities $\pi > 0$ expand the set of firms that are never liquidated (relative to the baseline model without refinancing opportunities, $\pi = 0$). As a result, better access to outside financing (i.e., higher π) also improves the access to inside financing provided by the insider, which ultimately helps to avert liquidation of a firm under financial distress. The reason is that better

[12] Simply totally differentiate (43) w.r.t. C and use $P''(C) < 0 < P'(C)$.

[13] The formal argument runs as follows. Evaluating the HJB (37) at $C = \overline{C}$ and imposing $\Delta = \alpha = \gamma = 0$ as raising any amount of cash would just lead to immediate payout of the same amount, we note that we are left with the same funding bound (i.e., $Y^A \geq 0$) as in our no-refinancing baseline model.

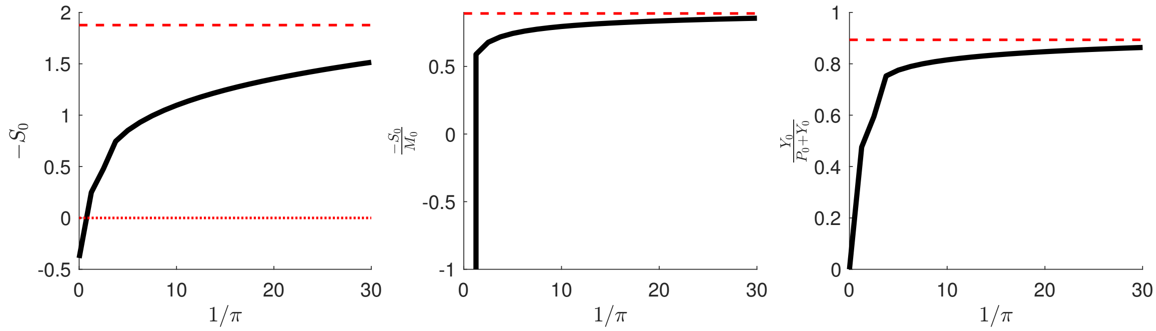

Figure 5: Model Outcomes at time zero w.r.t π. When $1/\pi$, then $-S_0/M_0 \to -\infty$.

access to outside financing facilitates more efficient risk-sharing with risk-neutral outside investors, thereby boosting the firm value and incentivizing the insider to keep the firm alive.

Note that differences to the *Second Best* and the *Baseline* arise once a firm finds its initial funding: (i) the set of firms that can attract funding without the future possibility of default expands, and (ii) even firms facing default do so at either lower likelihood of default and/or lower cost of holding cash, thus leading to higher ex-ante valuations compared to the *Baseline*. As π increases, we converge towards the *Second Best*, with both default and cash-holdings shrinking to zero.

Figure 5 illustrates the initial financing of the firm. On the horizontal axis we map $1/\pi$, the expected time between refinancing opportunities. The dashed red line in each of the panels is the reference *Baseline* value from Section 2. Let us first discuss the left panel, which is the initial cash contributed by the insider, $-S_0$. In the *Baseline*, the insider has to optimally co-invest with the outsider to maintain initial incentives, in which case $-S_0 > 0$. As refinancing oppportunities improve and $1/\pi$ decreases, we see that the insider is able to shrink his initial investment, until the refinancing market is so liquid that he eventually is able to take out extract cash, in which case $-S_0 < 0$. Related to this, the middle panel demonstrates that the fraction of initial cash contributed by the insider (i.e., $-S_0/M_0$) increases in $1/\pi$. Finally, consider the right panel depicting initial stake in the firm by the insider. As financial market access improves, the insider is able to reduce her initial stake in the firm.

5 Ownership dynamics and inside credit

This section endogenizes the choice of $\underline{C}$ for firms that are never liquidated, i.e., for firms with $Y^A > 0$. To highlight the key economic forces at play, we conduct the analysis for $\pi = 0$, that is,

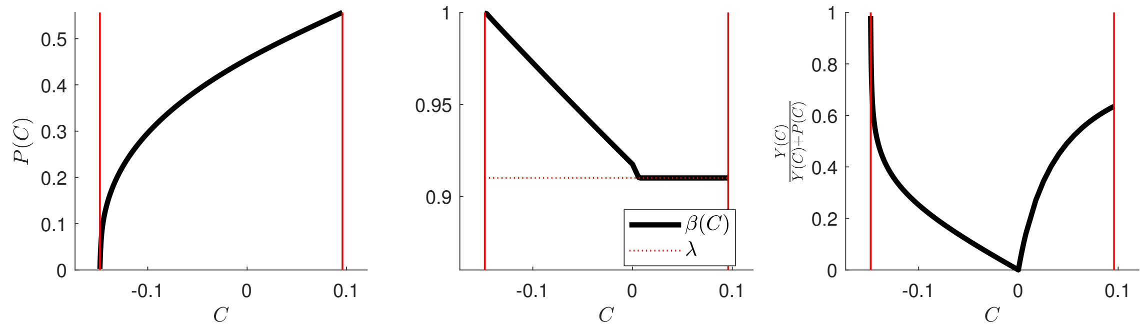

Figure 6: Numerical Example with endogenous lower boundary $\underline{C}$. The figure uses the baseline parameters, except that it uses a lower cash-flow volatility $\sigma = 0.95$ to ensure that $Y^A > 0 \Leftrightarrow \underline{C} < 0$.

when there are no refinancing opportunities after time zero. The Appendix discusses the general case with refinancing opportunities, $\pi \geq 0$.

To avert liquidation when cash reserves are exhausted and $M = 0$, the insider provides (inside) financing (i.e., inside credit). In exchange for the funds she injects into the firm, the insider receives additional deferred compensation Y and thus a larger ownership stake within the firm, leading to $Y > M = 0$ (i.e., $C < 0$). The maximal amount of inside credit is determined by the value of $\underline{C}$, which is endogenous and characterized below.

The firm's financial slack stems now from both accumulated cash and the inside credit capacity. In optimum, the firm finances its operations by drawing on its cash-reserves at certain times, while taking inside credit at other times. That is, $\overline{C} > 0 > \underline{C}$, with payout boundary $\overline{C}$. The firm holds cash $M > 0$ and retains earnings to accumulate more cash when $C > 0$, until the liquidity level reaches the dividend boundary $\overline{C}$. Whenever $C \in [\underline{C}, 0]$, the firm does not hold cash and financing is provided by the insider. Thus, $M = 0$ and $Y = -C$ and hence $dY = -dC$, whereby dC evolves according to (22). Intuitively, $Y = -C > 0$ measures the amount of inside credit outstanding and $C - \underline{C}$ captures the firm's remaining debt capacity. Also note that because cash M cannot become negative, it must be that $dM = 0$ and hence by means of (2), $dw = dX$. Thus, any earnings are paid out to the insider to reduce the amount of inside credit outstanding.

Naturally, the value of deferred compensation Y that can be credibly promised to the insider is bounded (due to $Y(C) + P(C) < \frac{\mu}{r}$ and $P(C)$ for $C \leq 0$), leading to the endogenous credit limit $Y(C) \leq -\underline{C} < \infty$ for $C \leq 0$. Notably, in optimum, this credit limit is equal to the firm's autarky value, in that $\underline{C} = -Y^A$. The intuition is that to credibly promise the insider's deferred compensation Y (i.e., the repayment of the inside credit), the firm serves as collateral for the inside credit. In turn, the insider is willing to provide inside credit up to her private valuation of the underlying collateral, which is given by the firm's autarky value Y^A.

Remarkably, the insider's deferred compensation $Y(C)$ and thus her firm ownership stake $\frac{Y(C)}{Y(C)+P(C)}$ serves a dual role. After good past performance (i.e., when $C > 0$), the firm holds cash $M > 0$ and deferred compensation is necessary to mitigate agency problems associated with free cash. After poor past performance (i.e., under financial distress), the free cash agency problem disappears due to $M = 0$, yet the insider holds a ownership stake (i.e., $Y > 0$) that serves as collateral for the inside credit she provides. As a result, the insider's ownership stake in the firm is high for high or low values of C (i.e., follows a V-shaped pattern in C), which is illustrated in Figure 6. When C reaches $\underline{C}$, the insider takes full control, in that her ownership stake grows to one for C close to $\underline{C}$. By taking control, the insider resolves financial distress. Indeed, the insider absorbs all cash-flow risk (i.e., $\beta(\underline{C}) = 1$) and the firm's net liquidity position gradually improves because the drift of C is strictly positive for all $C \geq \underline{C}$, thereby pulling the firm out of financial distress.

In contrast, for firms with $Y^A < 0$, the ownership dynamics look different. When $Y^A < 0$, the firm is liquidated once $C = 0$ and the insider does not provide inside credit. As Figure 1, the insider's ownership diminishes, when the firm is close to default, and increases following good performance. In addition, the insider never takes full control, in that her ownership stake in the firm is at any time strictly smaller than one. The following Proposition 4 summarizes the analytical findings of this section.

Proposition 4. *Suppose that $\underline{C}$ is endogenous. Then, the following holds:*

1. *In optimum, $\underline{C} = -\max\{Y^A, 0\}$ and $\overline{C} > 0$.*
2. *When $C < 0$, then $M(C) = 0$ and $Y(C) = -C$. When $C > 0$, then $Y(C) = \frac{\lambda C}{1-\lambda}$ and $M(C) = \frac{C}{1-\lambda}$*
3. *The outside equity value satisfies $\lim_{C\to\underline{C}} P(C) = 0$ and the insider's ownership stake satisfies $\lim_{C\to\underline{C}} \frac{Y(C)}{Y(C)+P(C)} = 1$.*

6 Conclusion

We present a model of liquidity management and financing decisions under moral hazard in which a firm accumulates cash to forestall liquidity default. When the cash balance is high, a tension arises between accumulating more cash to reduce the probability of default and providing incentives for the inside shareholder. When the cash balance is low, the firm hedges against liquidity default by transferring cash flow risk to the inside shareholder via high powered incentives. This risk transfer occurs even though the inside shareholder is risk averse and the outside shareholders are risk neutral because default is costly. Firms with more volatile cash flows transfer less risk to the inside shareholder and hold more cash. Agency conflicts lead to endogenous flotation costs related to the severity of the moral hazard problem, even in a market with no physical cost of raising financing. These flotation costs are state-dependent, lead to raising more than a static optimization would imply, and sometimes even lead to large cash-payouts to the agent in case of successful refinancing. Finally, because the inside shareholder absorbs part of the liquidity risk, the firm's stock return volatility can be non-monotonic in the level of cash and decreases in the severity of moral hazard.

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Appendix

A Parameters for the Numerical Analysis

We pick the parameters for the numerical analysis following prior contributions in the literature. We choose the agency parameter $\lambda = 0.91$, which is between 0.5 and 1 like in the numerical examples of DeMarzo and Sannikov (2006). Similar to He (2011), we pick the risk aversion parameter $\rho = 6$.

Following Bolton et al. (2011), we pick the mean cash-flow rate $\mu = 18\%$, the risk free rate $r = 6\%$, the carrying cost of holding cash $\delta = 1\%$ and the volatility of (scaled/adjusted) liquidity reserves equal to 9%. Notably, in our model the volatility of liquidity reserves $vol(dC)$ is time varying and state dependent. It can be shown that the state variable C is most of the time close to the payout boundary $\overline{C}$ (see e.g. Bolton et al. (2013)). Close to the payout boundary $\overline{C}$, our model implies a volatility of liquidity reserves:

$$vol(dC) \simeq (1 - \lambda)\sigma.$$

We therefore choose $\sigma = 1.025$ and $\lambda = 0.9$, so as to achieve $vol(dC) \simeq 9.3\%$, similar to Bolton et al. (2011). In case we allow for refinancing, we set $\pi = 0.25$.

These parameters also imply that $\mu < \rho r \sigma^2/2$ so that liquidation, $\tau < \infty$, is optimal under the baseline paramterization.

B Preliminaries

B.1 Regularity Conditions

Uncertainty is modelled via a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$, generated by X and N, i.e., $\mathcal{F}_t := \sigma(X_s : 0 \leq s \leq t)$. For any process $\mathcal{Y}$, adapted to $\mathbb{F}$, we also consider the left limit:

$$\mathcal{Y}_{t^-} := \lim_{s \uparrow t} \mathcal{Y}_s.$$

The left limit is formally needed as the cash diversion db_t can potentially induce a jump in cash holdings. The process $\{\mathcal{Y}_{t^-}\}$ is $\mathbb{F}$ predictable, i.e., adapted to:

$$\mathcal{F}_{t^-} := \lim_{s \uparrow t} \mathcal{F}_s.$$

We further write:

$$\mathbb{E}_x[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_x] \;\forall\; t \geq 0 \;\wedge\; x \in \{t, t^-\},$$

where $\mathbb{E}_x[\cdot] =: \mathbb{E}[\cdot]$ for $x \in \{0, 0^-\}$.

Throughout the paper and for all problems, we impose finite utility for any consumption process c:

$$\mathbb{E}\left[\int_0^t e^{-rs}|u(c_s)|ds\right] < \infty \;\forall\; t \geq 0$$

and square integrability conditions of dividend payouts Div and payments w:

$$\mathbb{E}\left[\int_0^t e^{-rs} dDiv_s\right]^2 < \infty \text{ and } \mathbb{E}\left[\int_0^t e^{-rs} dw_s\right]^2 < \infty \;\forall\; t \geq 0. \tag{B.1}$$

Finite utility implies that

$$\lim_{t\to\infty} e^{-rt}U_t(\cdot) \equiv \lim_{t\to\infty} e^{-rt}\mathbb{E}\left[\int_t^\infty e^{-r(s-t)}u(c_s)ds\right] = 0, \tag{B.2}$$

where $U_t(\cdot)$ represents the insider's continuation value under any, admissible strategy, suppressed for convenience. Condition (B.2) is also known as the *transversality condition* for the co-state, when solving the problem by means of Pontryagin's maximum principle (compare e.g. Williams (2015)).

Next, note that

$$\hat{S}_t = \int_0^t e^{r(t-s)}dw_s - \int_0^t e^{r(t-s)}\hat{c}_s ds + \hat{S}_0 e^{rt}$$

for the consumption process $\hat{c}$ specified by payout agreement $\mathcal{C}$, while c is the insider's actual consumption. Savings $\hat{S}$ corresponds to consumption $\hat{c}$ and savings S to consumption c.

We impose the no-Ponzi condition for all feasible consumption processes $c, \hat{c}$:

$$\mathbb{P}(\lim_{t\to\infty} e^{-rt}S_t \geq 0) = \mathbb{P}(\lim_{t\to\infty} e^{-rt}\hat{S}_t \geq 0) = 1.$$

Further, $c, \hat{c}$ must satisfy the transversality condition:

$$\lim_{t\to\infty} e^{-rt}\mathbb{E}u'(c_t)S_t = 0 = \lim_{t\to\infty} e^{-rt}\mathbb{E}u'(c_t)S_t = 0 \tag{B.3}$$

Due to finite utility it follows that marginal utility – which is proportional to flow utility – must also be finite $\mathbb{P}$-almost surely, so that one can disregard $u'(c_t)$ in the transversality condition, which leads to

$$\lim_{t\to\infty} e^{-rt}\mathbb{E}S_t = \lim_{t\to\infty} e^{-rt}\mathbb{E}\hat{S}_t = 0.$$

Combined with the no-Ponzi condition, it follows after invoking Fatou's Lemma in fact that

$$\mathbb{P}(\lim_{t\to\infty} e^{-rt}S_t = 0) = \mathbb{P}(\lim_{t\to\infty} e^{-rt}\hat{S}_t = 0) = 1,$$

which we refer to as the *transversality condition*, even though it emerges as a combination of transversality and No-Ponzi condition. By the triangle inequality:

$$\lim_{t\to\infty} e^{-rt}|S_t - \hat{S}_t| = 0 \ \mathbb{P} - a.s. \Longrightarrow \lim_{t\to\infty} e^{-rt}|\hat{c}_t - c_t| = 0 \ \mathbb{P} - a.s.$$

For technical reasons, we postulate that the process β is almost surely bounded, so that $|\beta_t| < M$ almost surely, i.e. $\mathbb{P}\big(|\beta_t| < M\big) = 1$ for any t. The equivalence of the measures $\mathbb{P}, \mathbb{P}^b$ (to be discussed in the next paragraph) ensures that the sensitivities are almost surely bounded under each probability measure used throughout the paper. We assume $M \in \mathbb{R}_+$ to be sufficiently large, so that this imposed constraint actually never binds in optimum.

B.2 Change of Measure

To start with, fix a probability measure $\mathbb{P}$, such that $dX_t = \mu dt + \sigma dZ_t$ with a $\mathbb{F}$-progressive standard Brownian Motion Z under the measure $\mathbb{P}$. Take a progressive process b, that is absolutely continuous and one can write $db_t = b_t^0 dt$ for some process b^0. Define the process χ via $\chi_t = \frac{db_t}{\sigma dt}$ for all $t \geq 0$, almost surely. Further, let

$$\xi_t = \xi_t(b) = \exp\Big(\int_0^t \chi_u dZ_u - \frac{1}{2}\int_0^t \chi_u^2 du\Big).$$

Assuming that the so-called Novikov condition is satisfied, i.e.,

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^\tau \chi_t^2 dt\right)\right] < \infty,$$

it follows that ξ follows a martingale. Given our restriction of bounded sensitivities, the Novikov-Condition is evidently met. Due to $\mathbb{E}[\xi_0] = 1$, it is evident that ξ is a progressive density process and defines a probability measure $\mathbb{P}^b$ via the Radon-Nikodym derivative

$$\left(\frac{d\mathbb{P}^b}{d\mathbb{P}}\right)\bigg|_{\mathcal{F}_t} = \xi_t = \xi_t(b).$$

Under the probability measure $\mathbb{P}^b$, the process Z^b with

$$Z_t^b = Z_t - \int_0^t \chi_u du = \frac{X_t - \mu t - \int_0^t b_u^0 du}{\sigma}$$

follows a standard Brownian Motion up to the stopping time τ. All measures $\{\mathbb{P}, \mathbb{P}^b : b\}$ are equivalent for suitable absolutely continuous processes b, that satisfy the above stated conditions, such that the measures share the same null sets.

Girsanov's theorem is only applicable if b is absolutely continuous $\mathbb{P}$-almost surely, in which case Z^b follows a Brownian Motion under $\mathbb{P}^b$.

C Proof of Proposition 1

We split up the proof into parts. First, we establish the representation of U by means of a stochastic differential equation, given a payout agreement $\mathcal{C}$. From there, we proceed to show the claim regarding incentive compatibility.

C.1 Martingale Representation: Proof of Proposition 1 Part (1)

Proof. Let in the following $\mathcal{C} = (Div, w)$ represent the payout agreement, which — by assumption — induces $b_t = 0$ almost surely for all $t \geq 0$. We denote the insider's continuation value by

$$U_t = U_t(\mathcal{C}) = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}u(c_s))ds\right].$$

Define

$$A_t \equiv \mathbb{E}_t\left[\int_0^\infty e^{-rt}u(c_s)ds\right] = \int_0^t e^{-rs}u(c_s)ds + e^{-rt}U_t(\mathcal{C}) \tag{C.1}$$

By construction, $\{A_t : 0 \leq t \leq \infty\}$ is a square integrable martingale, progressive with respect to $\mathbb{F}$ under $\mathbb{P}$. By the martingale representation theorem, there exist now $\mathbb{F}$-predictable processes β such that

$$e^{rt}dA_t = (-\rho r U_{t-})\beta_t\big(dX_t - \mu dt\big)$$

and therefore

$$dU_t = rU_{t^-}dt - u(c_t)dt + (-\rho rU_{t^-})\beta_t\big(dX_t - \mu dt\big).$$

Having derived an SDE for U, we can use Ito's Lemma to derive the law of motion of

$$W_t = W(U_t) := \frac{-\ln(-\rho rU_t)}{\rho r}.$$

Direct calculations yield:

$$dW_t = \frac{\rho r(\beta_t\sigma^2)}{2}dt + \beta_t(dX_t - \mu dt) + \frac{u(c_t) - rU_{t^-}}{\rho rU_{t^-}}\, dt.$$

$\square$

C.2 Proof of Proposition 1 Part (2)

The claim follows from the below Lemma. Define $Y := W - S$.

Lemma 2.

$$Y_t = W_t - S_t = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}\left(dw_s - \frac{\rho r}{2}(\beta_s\sigma)^2ds\right)\right].$$

Proof. Take the SDEs:

$$dS = rSdt + dw - cdt, \quad c = rW$$
$$dW = \frac{\rho r}{2}(\beta\sigma)^2dt + \beta\sigma dZ.$$

which — by our imposed regularity conditions and assumptions — admit unique strong solutions S, W under the measure $\mathbb{P}$. Calculate:

$$dY = dW - dS = \frac{\rho r}{2}(\beta\sigma)^2dt + \beta\sigma dZ - dw + rYdt.$$

On the other hand, take

$$Y_t = \mathbb{E}_t\left[\int_t^\infty e^{-r(s-t)}\left(dw_s - \frac{\rho r}{2}(\beta_s\sigma)^2ds\right)\right].$$

By mimicking the arguments presented in the proof of Proposition (1) i), one can show that by the martingale representation theorem, there exists a unique $\mathbb{F}$ progressive process γ such that:

$$dY = \frac{\rho r}{2}(\beta\sigma)^2dt + \gamma\sigma dZ - dw + rYdt.$$

Uniqueness (up to null sets) implies $\gamma = \beta$ almost surely, which ends the proof. $\square$

C.3 Incentive Compatibility: Proof of Proposition 1 Parts (3) and (4)

The proof proceeds in several steps. Step I presents a Lemma that characterizes the insider's consumption smoothing. Step II introduces additional notation and puts structure to the problem. Step III analyzes lumpy cash diversions. Step IV analyzes infinitesimal cash (flow) diversions of order dt Step V verifies that local optimality conditions ensure global optimality.

Step I. Here, we state an auxiliary Lemma:

Proof. We prove first the following auxiliary Lemma

Lemma 3. *Fix a $\mathbb{F}$-predictable process $\hat{c}$ and let $\mathcal{S} \in \mathbb{R}$. Consider the problem*

$$U_t = \max_{\{c_s\}_{s\geq t}} \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} u(c_s) ds \right]$$

$$\text{subject to } dS_s = rS_s ds + d\hat{c}_s ds - c_s ds, S_t = 0 \text{ and } \lim_{s\to\infty} e^{-r(s-t)}|S_t - S_s| = 0 \text{ a.s.}$$

Next consider the problem

$$U'_t = \max_{\{\tilde{c}_s\}_{s\geq t}} \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} u(\tilde{c}_s) ds \right]$$

$$\text{subject to } dS_s = rS_s ds + d\hat{c}_s ds - \tilde{c}_s ds, S_t = \mathcal{S} \text{ and } \lim_{s\to\infty} e^{-r(s-t)}|S - S_s| = 0 \text{ a.s.}$$

Then, $c_t + r\mathcal{S} = \tilde{c}_t$ and $U'_t = e^{-\rho r \mathcal{S}} U_t$.

Proof. Suppose that there exists a process $c' \neq \tilde{c}$, which satisfies the transversality condition $\lim_{s\to\infty} e^{-r(s-t)}|S_t - S_s| = 0$ a.s., such that

$$U'_t(c') > U'_t(\tilde{c}) = e^{-\rho r \mathcal{S}} U_t.$$

Define the process c'' via $c''_t = c'_t - r\mathcal{S}$. Then c'' satisfies the transversality condition and

$$\mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} u(c''_s) ds \right] = e^{\rho r \mathcal{S}} U'_t(\{c'\}) > U_t,$$

a contradiction. □

Next, we characterize the insider's private consumption c and provide necessary and sufficient conditions for $\mathcal{C}$ to be incentive-compatible, in that $\hat{b}_t = b_t = 0$ for all $t \geq 0$ holds almost surely.

Step II. Write, $db_t = (b_t^0 - b_t^2)dt + db_t^1$, where b^0 and b^2 are absolutely continuous and almost surely positive, i.e, write $db_t = \hat{b}_t dt + db_t^1$, where $b_t^0 = \max\{0, \hat{b}_t\}$ and $b_t^2 = -\min\{0, \hat{b}_t\}$. Here, b^0 corresponds to cash-flow diverted, while b^2 is the amount by which cash-flow is boosted by means of the insider's savings account. In the following, a "hat" above a quantity denotes that this is the quantity under the zero diversion policy. That is, $\hat{U}_t$ is the continatuon value of the insider that never diverts, in that $b_t = 0$ for all t almost surely. By contrast, U_t is the continuation value under some arbitrary diversion b. The proof is complete if we provide conditions that ensure $U_t = \hat{U}_t$ almost surely for all t.

Let $\hat{S}_t$ the savings under the truthful policy, $b_t = 0 \; \forall \, t \geq 0$ and S_t the savings under a potentially different diversion policy b. Define $\Delta_t \equiv S_t - \hat{S}_t$ the deviation state with $\Delta_0 = 0$ and note that

$$d\Delta_t = r\Delta_t dt + \lambda b_t^0 dt + \lambda db_t^1 - b_t^2 dt,$$

where $\hat{c}$ is such that $S_t = \hat{S}_t$ under $b_t = 0 \; \forall \; t \geq 0$, i.e. $\Delta_t = 0$ for all t, and c the consumption otherwise under a general process b.

Let U the insider's actual continuation value, so that

$$U_t(c) = U_t \equiv \mathbb{E}_t^b \left[\int_t^\infty e^{-r(s-t)} u(c_s) ds \right],$$

where the expectation $\mathbb{E}_t^b$ is taken under the measure $\mathbb{P}^b$, induced by the choice of b.

We rewrite for $t < \tau$:

$$dU_t = rU_{t^-}dt - u(c_t)dt + (-\rho r U_{t^-})\beta_t \big(dZ_t^b + b_t^0 dt - b_t^2 dt\big).$$

Note that $dZ_t^b \equiv (dX_t - \mu dt + b_t^0 dt - b_t^2 dt)/\sigma$ is the increment of a standard Brownian Motion under the measure $\mathbb{P}^b$. Define the insider's (actual) certainty equivalent $W_t = \frac{-\ln(-\rho r U_t)}{\rho r}$ and $Y_t \equiv W_t - S_t$. Differently, the continuation value $\hat{U}$ under the truthful policy (zero diversion) follows:

$$d\hat{U}_t = r\hat{U}_{t^-}dt - u(\hat{c}_t)dt + (-\rho r \hat{U}_{t^-})\hat{\beta}_t dZ_t,$$

where Z is a standard Brownian motion under the measure induced by zero diversion.

Step III. First, let us consider the insider deviates at time t^- through specifying $M_{t^-} \geq db_t^1 > 0$, so that $db_t \notin o(dt)$. $S_t = S_{t^-} + \lambda db_t^1$, yielding continuation value by Lemma 3:

$$\tilde{U}_t e^{-\rho r \lambda db_t^1},$$

where $\tilde{U}_t$ is the continuation value stipulated by the optimal payout agreement upon a deviation db_t^1. The deviation is not profitable, if and only if

$$\hat{U}_t e^{-\rho r \lambda db_t^1} \leq U_{t^-}$$

On the other hand, recall the insider can always leave the firm and realize continuation utility of at least

$$U_{t^-} e^{\rho r Y_{t^-}}$$

in all scenarios. In particular, punishment is limited in that:

$$\tilde{U}_t \geq U_{t^-} e^{\rho r Y_{t^-}}$$

Hence, incentives are optimized for

$$\tilde{U}_t = U_{t^-} e^{\rho r Y_{t^-}},$$

leading to the incentive condition

$$U_{t^-} e^{-\rho r \lambda db_t^1} e^{\rho r Y_{t^-}} \leq U_{t^-},$$

ruling out $db_t^1 > 0$. This is equivalent to:

$$Y_{t^-} \geq \lambda db_t^1$$

and must hold for all $db_t^1 \in [0, M_{t^-}]$. Hence, a necessary condition for the payout agreement $\mathcal{C}$ to be incentive compatible is that $Y_{t^-} \geq \lambda M_{t^-}$ with probability one for all times $t \geq 0$.

Step IV. Let us turn to strategies where $db^1_t = 0$ for all $t \geq 0$. Let $t > 0$ and suppose the insider follows the truthful diversion policy from time t onwards, in that $b^0_s = 0$ for all $s \geq t$. Also, Lemma 3 readily implies that and $c_s = \hat{c}_s + r\Delta_t$ for $s \geq t$. The payoff from following this strategy is represented by the auxiliary gain process

$$G^M_t \equiv G^M_t(c,b) = \int_0^t e^{-rs}u(c_s)ds + e^{-\rho r\Delta_t}e^{-rt}\hat{U}_t$$

and by means of Lemma 3, it suffices to consider deviations of this type, which yield weakly higher payoff than deviations of any other type. In addition, $U_s = e^{-\rho r\Delta_t}\hat{U}_s$ for $s \geq t$, so that

$$G^M_t \equiv G^M_t(c,b) = \int_0^t e^{-rs}u(c_s)ds + e^{-rt}U_t$$

Next, note that the transversality condition and finite utility imply that $|e^{-\rho r\Delta}U_t| < \infty$ almost surely for all $t \geq 0$, so that $\lim_{t\to\infty}\mathbb{E}^b e^{-\rho r\Delta_t}e^{-rt}U_t = 0$ for any possible strategy of the insider. This implies that the insider's actual payoff equals

$$U_{0^-} = \max_{c,b^0}\mathbb{E}^b\int_0^\infty e^{-rs}u(c_s)ds = \max_{c,b^0}\mathbb{E}^b G^M_\infty = \max_{c,b^0}\mathbb{E}^b\lim_{t\to\infty}G^M_t.$$

By Itô's Lemma:

$$\begin{aligned}
&e^{\rho r\Delta_t}e^{rt}dG^M_t\\
&= \Big(u(c_t)e^{\rho r\Delta_t} - u(\hat{c}_t) - \rho r\hat{U}_{t^-}(r\Delta_t + \hat{c}_t - c_t + \lambda b^0_t - b^2_t) - (-\rho r\hat{U}_{t^-})\hat{\beta}_t(b^0_t - b^2_t)\Big)dt\\
&+ (-\rho r\hat{U}_{t^-})\hat{\beta}_t dZ^b_t\\
&\equiv \mu^M_{tG}(\cdot)dt + (-\rho r\hat{U}_{t^-})\hat{\beta}_t dZ^b_t
\end{aligned}$$

Observe that, because $\hat{\beta}$ is bounded and finite utility is imposed, we have

$$\mathbb{E}^b\bigg(\int_0^t e^{-rs}\beta_s(-\rho r\hat{U}_{s^-})dZ^b_s\bigg) = 0,$$

for any absolutely continuous b. It is then evident that by choosing $b^0_t = b^2_t = 0$, $c_t = \hat{c}_t$, the insider can ensure that $\Delta_t = \mu^M_{tG}(\cdot) = 0$ for all $t \geq 0$, in which case $\{G^M(\hat{c},0)\}$ follows a martingale under $\mathbb{P}$ with last element $G^M_\infty(\cdot)$, such that $\mathbb{E}|G^M_\infty(\hat{c},0)| < \infty$ due to the regularity conditions we impose. Hence, by the optional sampling theorem:

$$U_{0^-} = \max_{c,b^0}\mathbb{E}^b G^M_\infty(c,b^0) \geq \mathbb{E}G^M_\infty(\hat{c},0) = \lim_{t\to\infty}\mathbb{E}G^M_t(\hat{c},0) = \hat{U}_{0^-}.$$

Next, observe that the highest value that $\mu^M_{tG}(\cdot)$ can obtain given Δ_t is given by the maximization over c_t and b^0_t, where the solution satisfies the following FOC:

$$u'(c_t)e^{r\rho\Delta_t} = -\rho r\hat{U}_{t^-},$$

which implies

$$u(c_t + r\Delta_t) = r\hat{U}_{t^-},$$

and $b_t^0 = b_t^2 = 0$ if and only if:

$$\lambda \Delta \rho r u(c_t) e^{\rho r \Delta_t} - \lambda \rho r \hat{U}_{t-} + (\rho r \hat{U}_{t-})\hat{\beta}_t \leq 0 \text{ and}$$
$$\Delta \rho r u(c_t) e^{\rho r \Delta_t} + \rho r \hat{U}_{t-} + (\rho r \hat{U}_{t-})\hat{\beta}_t \leq 0.$$

If $\mathcal{C}$ is such that $r\hat{U}_{t-}e^{-\rho r \Delta_t} = u(\hat{c}_t)$ and $1 \geq \hat{\beta}_t \geq \lambda$ hold for all $t \geq 0$, it follows $c_t = \hat{c}_t$ and $b_t^0 = b_t^1 = 0$ for all $t \geq 0$, in which case $\Delta_t = \mu_{tG}^M(\cdot) = 0$. Indeed, because the deviation gains are concave in the state Δ, the first order conditions are sufficient.

Step V. Hence, any other strategy tuple (c, b^0) makes the process $G^M(c, b^0)$ a supermartingale under the measure $\mathbb{P}^b$, i.e.

$$\hat{U}_{0-} = G_0^M(\hat{c}, 0) \geq \mathbb{E}^b G_t^M(c, b^0)$$

Because our regularity conditions ensure that $G^M(c, b^0)$ is bounded from below, we can thus take limits on both sides and apply optional sampling to obtain

$$\hat{U}_{0-} \geq \lim_{t\to\infty} \mathbb{E}^b G_t^M(c, b^0) = \mathbb{E}^b \lim_{t\to\infty} G_t^M(c, b^0) = \mathbb{E}^b G_\infty^M(c, b^0)$$

and in particular

$$\hat{U}_{0-} \geq \max_{c, b^0} \mathbb{E}^b G_\infty^M(c, b^0) = U_{0-}.$$

While we focused on strategies $(c, 0, b^0)$ and $(\hat{c}, b^1, 0)$ separately, note that

$$U_{0-} \geq \max_{c, b^0} \mathbb{E}^b G_\infty^M(c, 0, b^0) \text{ and } U_{0-} \geq \max_{b^1} \mathbb{E}^b G_\infty^M(\hat{c}, b^1, 0) \Longrightarrow U_{0-} \geq \max_{c, b^1, b^0} \mathbb{E}^b G_\infty^M(c, b^1, b^0).$$

This is because the maximal utility the insider can obtain at time t equals $e^{-\rho r \Delta_t} U_{t-}$ under any consumption c, while the deviation utility is given by

$$e^{-\rho r(\Delta_t + \lambda d b_t^1)} \hat{U}_t$$

which is smaller than $e^{-\rho r \Delta_t} U_{t-}$ if and only if $Y_{t-} \geq \lambda M_{t-}$ by Step II.

Therefore, $U_{0-} = \hat{U}_{0-}$ and $(c_t, b_t) = (\hat{c}_t, 0)$ for all $t \geq 0$ is the optimal strategy for the insider if and only if $1 \geq \hat{\beta}_t \geq \lambda, rU_{t-} = u(\hat{c}_t), Y_{t-} \geq \lambda M_{t-}$ are satisfied for all $t \geq 0$ with probability one. In this case, the payout agreement $\mathcal{C}$ is incentive compatible and the consumption path as desired. In particular, $\hat{\beta} = \beta$ and $\hat{U} = U$ almost surely, which concludes the proof. □

D The Outside Equity Value: Proof of Proposition 2

The following Section aims to maximize the value of the outside equity claim:

$$P_0 = \mathbb{E}\left[\int_0^\tau e^{-rt} dDiv_t + e^{-r\tau} L\right],$$

over all incentive compatible payout agreements $\mathcal{C}$. Here, the endogeneous stopping time

$$\tau = \inf\{t \geq 0 : dDiv_s = 0 \;\text{ a.s. } \;\forall s \geq t\}$$

is the "liquidation/termination" time. In equilibrium, τ is adapted to $\mathbb{F}$. The proof follows a guess and verify approach.

First, we conjecture the shape of the optimal payout agreement and obtain that the outside equity value under the optimal payout agreement solves a partial differential equation with three states (M, W, S) on an endogeneous state space.

Second, we show that given this conjecture, we can reduce the dimensionality of the state space and show that the outside equity value solves a ordinary differential equation with state C. We also show that the state space is bounded.

Third, we verify that the conjectured shape of the payout agreement is indeed optimal.

D.1 Part I

Let $\hat{P}$ the outside equity value under the optimal payout agreement $\mathcal{C}$. Per se, the endogeneous state space $\mathcal{M}^*$ is three dimensional and we have to keep track of three state M, W, S. Given shareholders' risk-neutrality, it is natural to conjecture that payouts Div follow a bang-bang policy and reflect states into the interior of the state space $\mathcal{M}^*$.

Thus, by the dynamic programming principle, $\hat{P}$ must solve the PDE

$$r\hat{P}\left(W, M, S\right) dt = \max_{\beta \geq \lambda, dw \in \mathcal{I}, dDiv \geq 0} dDiv + \mathbb{E}[d\hat{P}(M, W, S)] \tag{D.1}$$

in the interior of the state space $\mathcal{M}^*$. To save on notation we write the HJB-equation in differential form. The term $d\hat{P}(M, W, S)$ can be expanded by Ito's Lemma. Here, wages dw are subject to an arbitrary constraint $\mathcal{I}$. Thus, in the interior of $\mathcal{M}^*$ the following HJB-equation holds:

$$r\hat{P}\left(W, M, S\right) = \max_{\beta \geq \lambda, dw \in \mathcal{I}} \frac{\mathbb{E}[d\hat{P}(M, W, S)]}{dt}. \tag{D.2}$$

It is beyond the scope of the paper to provide a formal existence and uniqueness proof for a solution to (D.2). Therefore, we assume throughout the remainder:

Assumption 2 (Existence, Uniqueness & Smoothness)**.** *The PDE (D.2) admits a unique solution $\hat{P} \in C^2$. The solution $\hat{P}$ is twice continously differentiable.*

Assumption 1 implies that — owing to sufficient smoothness — (D.2) also holds on the edges of $\mathcal{M}^*$ and so holds on the entire state space.

D.2 Part II

Due to the absence of wealth effects, one can show that the value function takes the form $P(M, Y) = P(M, W - S) = \hat{P}(M, W, S)$. This relationship is established in the below Lemma:

Lemma 4. *Let $\hat{P}(M, W, S) \in C^2$ the outside equity value under the optimal payout agreement $\mathcal{C}$, solving (D.2). Then $\hat{P}(M, W, S) = P(M, W - S) = P(M, Y)$ for some function $P \in C^2$ on the state space.*

Proof. The proof proceeds by a guess and verify approach. Let us merely conjecture $\hat{P}(M, W, S) = P(M, Y)$ and show that such a function solves (D.2). By postulated uniqueness, this completes the proof.

In the following, a subscript denotes the partial derivative wrt. to the respective variable and we omit the argument of the function $\hat{P}(\cdot)$ to avoid clutter. Note that

$$\hat{P}_W = -\hat{P}_S = P_Y. \tag{D.3}$$

As this is an identity in the interior of the state space $\mathcal{M}$, we can differentiate, so as to obtain:

$$\hat{P}_{WS} = -\hat{P}_{SS} = -P_{YY} \tag{D.4}$$
$$\hat{P}_{WW} = -\hat{P}_{WS} = P_{YY}. \tag{D.5}$$

To get a better overview, let us review the SDEs that determine the law of motion of (M, W, S) under an incentive compatible payout plan and in the interior of the state space, i.e., $db = dDiv = 0$:

$$\begin{aligned}
dM &= ((r-\delta)M + \mu)dt + \sigma dZ - dw \\
dS &= rSdt + dw - cdt, \quad c = rW \\
dW &= \frac{\rho r}{2}(\beta\sigma)^2 dt + \beta\sigma dZ \\
dY &= dW - dS = \frac{\rho r}{2}(\beta\sigma)^2 dt + \beta\sigma dZ - dw + rYdt.
\end{aligned}$$

Rewriting:

$$dS = -rYdt + dw$$

and expanding in (D.2), evaluated under the optimal controls, the term $\mathbb{E}d\hat{P}$ by virtue of Ito's Lemma yields:

$$\begin{aligned}
r\hat{P} &= \hat{P}_W\left[\frac{\rho r}{2}(\beta\sigma)^2\right] + \hat{P}_S\left[-rY + \frac{\mathbb{E}dw}{dt}\right] + \hat{P}_M\left[(r-\delta)M + \mu - \frac{\mathbb{E}dw}{dt}\right] \\
&+ \frac{1}{2} \times \left\{\hat{P}_{WW}(\beta\sigma)^2 + \hat{P}_{SS}\frac{< dw, dw >}{dt} + \hat{P}_{MM}\left[\sigma^2 - \frac{\sigma < dw, dZ >}{dt}\right]\right\} \\
&+ \hat{P}_{MS}\frac{- < dw, dw > + \sigma < dZ, dw >}{dt} + \hat{P}_{MW}\beta\sigma^2 + \hat{P}_{WS}\frac{< dw, dZ > \beta\sigma}{dt},
\end{aligned}$$

where $< \cdot, \cdot >$ denotes the quadratic variation of two processes (e.g., $< dZ, dZ >= dt$).

Note that controls dw and β depend on the value function and its derivatives by the HJB-equation (D.2). By the hypothesis $\hat{P}(M, W, S) = P(M, Y)$, it follows that dw and β therefore only depend on (M, Y). After substituting higher order derivatives of $\hat{P}$ by higher order derivatives of V by means of (D.3) and (D.4), one observes that the right-hand-side of the HJB-equation does not depend on (W, S) but on Y only. Thus, also the left-hand-side also is only a function of (M, Y) only, thereby confirming the guess that P solves (D.2), which concludes the proof. □

Thus, the outside equity value solves:

$$\begin{aligned}
rP = \max_{dw \leq M, \beta \geq \lambda} &\left\{ P_M\Big(\mu + (r-\delta)M - \frac{\mathbb{E}dw}{dt}\Big) + \frac{P_{MM}}{2}\left[\sigma^2 + \frac{< dw, dw > -\sigma < dw, dZ >}{dt}\right]\right. \\
&+ P_Y\Big(rY + \frac{\rho r}{2}(\beta\sigma)^2 - \frac{\mathbb{E}dw}{dt}\Big) + \frac{P_{YY}}{2}\left[\frac{< dw, dw > -\beta\sigma < dw, dZ >}{dt} + (\beta\sigma)^2\right] \\
&\left. + P_{MY}\left(\frac{< dw, dw > -(\beta+1)\sigma < dw, dZ >}{dt} + \beta\sigma^2\right)\right\}
\end{aligned} \tag{D.6}$$

In the following, we show that the state space is in fact a one-dimensional manifold, i.e., is one-dimensional.

D.3 Part III

Let us define $C := M - Y$ and rotate the state space $\mathcal{M} = \{(M,Y) : \exists\ t \ \text{ s.t. } \mathbb{P}((M_t, Y_t) = (M,Y) > 0\}$ to $\mathcal{M}^* = \{(C,Y) : \exists\ t \ \text{ s.t. } \mathbb{P}((C_t, Y_t) = (C,Y) > 0\}$. Clearly, on the interior of $\mathcal{M}^*$, dividend payouts are zero. By the previous part, with slight abuse of notation, there exists a function $P(C,Y)$ that equals the outside equity value at time $t \geq 0$ with $(C_t = C, Y_t = Y)$.

Define for any point (C,Y):

$$\varphi = \varphi(C,Y) = \frac{Y}{M}$$

One can readily calculate that C solves the SDE:

$$dC = \left(r - \frac{\delta}{1-\varphi}\right) C dt - \frac{\rho r}{2}(\beta\sigma)^2 dt + \mu dt + (1-\beta)\sigma dZ - dDiv. \tag{D.7}$$

Therefore, the outside equity value solves therefore:

$$\begin{aligned} rP = \max_{\varphi,\beta\geq\lambda,dw\leq M} \Bigg\{ & P_C\left(\left(r - \frac{\delta}{1-\varphi}\right) C - \frac{\rho r}{2}(\beta\sigma)^2 + \mu\right) + \frac{\sigma^2(1-\beta)^2}{2} P_{CC} \\ & + P_Y\left(rY + \frac{\rho r}{2}(\beta\sigma)^2 dt - \frac{\mathbb{E}dw}{dt}\right) \\ & + \frac{P_{YY}}{2}\left[\frac{< dw, dw > -\beta\sigma < dw, dZ >}{dt} + (\beta\sigma)^2\right] + P_{CY}\frac{\sigma(1-\beta) < dw, dZ >}{dt}\Bigg\}. \end{aligned} \tag{D.8}$$

It also follows that $P \in C^2$.

The following Lemma demonstrates that this function only depends on C.

Lemma 5. *The function P solving the PDE (D.8) only depends on C.*

Proof. Define for any point (C,Y):

$$\varphi = \varphi(C,Y) = \frac{Y}{M}$$

and note that incentive compatibility requires $\varphi \geq \lambda$. For any interior point $(C,Y) \in \mathcal{M}^*$ with $Y > \lambda M$, the firm's value function is given by (D.8). In the interior, $Y > \lambda M$ so that one can move by setting $dw = \varepsilon$ for $|\varepsilon| > 0$ sufficiently small implying that:

$$P(C, Y - \varepsilon) = P(C,Y)$$

for $|\varepsilon| > 0$ sufficiently small. Hence:

$$P_Y = 0$$

for all $(C,Y) \in int(\mathcal{M}^*)$ which readily implies:

$$P_{CY} = P_{YY} = 0$$

on $int(\mathcal{M}^*)$. By continuity:

$$P_Y = P_{CY} = P_{YY} = 0 \ \forall\ (C,Y) \in \mathcal{M}^*.$$

Thus, (D.8) becomes

$$rP = \max_{\beta\geq\lambda,\varphi\geq\lambda} \left\{ P_C\left(\left(r - \frac{\delta}{1-\varphi}\right) C - \frac{\rho r}{2}(\beta\sigma)^2 + \mu\right) + \frac{\sigma^2(1-\beta)^2}{2} P_{CC}\right\}, \tag{D.9}$$

Assume that $\varphi(\cdot), \beta(\cdot)$ only depend on C, i.e., is are function of C. Then, the RHS of (D.9) only depends on C and so does P. Conversely, if P only depends on C its derivatives do so as well. In this case, β, φ are functions of C. As a result, P only depends on C, thereby concluding the proof. $\square$

D.4 Part IV

The next Lemma shows that we have $M_t \geq Y_t$ with certainty in case $C_0 \geq 0$. Thus, given a starting value $C_0 \geq 0$ the set of states reached with positive probability is also contained in the non-negative real line $\mathbb{R}_+$ in that $\mathbb{P}(C_t \geq 0 \; \forall t \geq 0 | C_0 \geq 0) = 1$.

Lemma 6. *If $C_0 \geq 0$, then $\mathbb{P}(\{Y > M\} = \mathbb{P}(\{C < 0\} = 0$.*

Proof. Suppose that at time t, the state is $C_t = C > 0$, implying that $M_t > Y_t \geq 0$, as $Y_t < 0$ violates incentive compatibility. Then, the outside equity value solves the HJB-equation

$$rP = \max_{\beta \geq \lambda, \varphi \geq \lambda} \left\{ P_C \left(\left(r - \frac{\delta}{1-\varphi} \right) C - \frac{\rho r}{2} (\beta\sigma)^2 dt + \mu \right) + \frac{\sigma^2 (1-\beta)^2}{2} P_{CC} \right\},$$

and note that dividend payouts $dDiv > 0$ are always possible but not necessarily optimal, implying that $P_C > 0$. Hence, optimally $\varphi = \lambda$, so that for all $C > 0$:

$$Y = \lambda M < M.$$

Next, recall that by hypothesis $C_0 \geq 0$ implying that $M_0 \geq Y_0 \geq 0$. Define $t_0^i = \inf\{t \geq t_0^{i-1} : M_{t^-} = 0\}$ recursively as the i'th time cash-holdings M are equal to zero. Set $t_0^0 = 0$ and note that due to $C_0 > 0$ it must be that $\mathbb{P}(t_0^1 > 0) = 1$.

Recall that we have shown that $Y_t = \lambda M_t < M_t$ whenever $C_t > 0$. That is, $Y_t = \lambda M_t < M_t$ for all $t < t_0^1$ so that by continuity $Y_t = M_t = 0$ at time t_0^1, i.e., $t_0^1 = \inf\{t \geq 0 : C_{t^-} = 0\}$. Two scenarios are possible at time $t = t_0^1$.

First, the firm is liquidated, in which case $\tau = t_0^1 = t_0^i \; \forall \; i \geq 1$ and $dX_t = dM_t = dY_t = 0$ for all $t \geq t_0^1$, as desired. Liquidation yields at least outside equity value zero, $P_\tau \geq 0$.

Second, the firm is not liquidated in that $\tau > t_0^1$. As M_t cannot fall below zero, it must be at time $t = t_0^1$ that:

$$vol(dM_t) = 0 \Longrightarrow vol(dw_t) = \sigma. \tag{D.10}$$

As Y_t cannot fall below zero, it must be at time $t = t_0^1$ that:

$$vol(dY_t) = \beta_t\sigma - vol(dw_t) = 0 \Longrightarrow \beta_t = 1. \tag{D.11}$$

Hence, at time $t = t_0^1$, we have that:

$$\frac{dC_t}{dt} = \mu - \frac{\rho r \sigma^2}{2}.$$

The firm value in this scenario is given by:

$$P = \frac{1}{r} P_C \frac{dC}{dt},$$

which must be greater than zero, as liquidation is sub-optimal. Owing to $P_C > 0$, it follows that $dC/dt \geq 0$ at $t = t_0^1$, so that C_t does not fall below zero. A standard induction argument wrt. to $i \geq 1$ implies the assertion $\mathbb{P}(\{C < 0\}) = \mathbb{P}(\{Y > M\}) = 0$. This concludes the proof. $\square$

Lastly, we demonstrate that $C_0 \geq 0$.

Lemma 7. $C_0 \geq 0$.

Proof. The proof is an accounting exercise. Outside investors contribute $P_0 \geq 0$ dollars at inception. The insider contributes $Y_0 - d$ dollars and has continuation value (in monetary units) Y_0, with values $Y_0 \geq 0$ and d to be determined in optimum. Thus, initial cash holdings equal:

$$M_0 = P_0 + Y_0 - d$$

so that

$$C_0 = M_0 - Y_0 = P_0 - d.$$

The insider's payoff (in monetary units) equals:

$$Y_0 - (Y_0 - d) = d = P_0 - C_0.$$

By our previous steps, $P_0 = P(C_0)$ and because dividend payouts are always possible but not necessarily optimal, it must be that

$$P(C - \Delta) + \Delta \leq P(C)$$

for $\Delta > 0$ sufficiently small on the interior of $\mathcal{M}^*$. Hence: $P'(C) \geq 1$. Thus:

$$\frac{\partial d}{\partial C_0} = P'(C_0) - 1 \geq 0,$$

so that $C_0 \geq 0$ is optimal. □

Hence, we have shown that $C_t \geq 0$ for all $t \geq 0$.

D.4.1 Part V

Initially, in Part II, we have conjectured that the optimal dividend policy reflects back into the interior of the state space. Given that the space is one dimensional (see Part III and Part IV) and bounded below from zero (see Part V), dividend payouts Div take the form $dDiv = \max\{C - \overline{C}, 0\}$. At the reflecting boundary, the standard smooth pasting and super contact conditions must hold in that

$$P'(\overline{C}) - 1 = P''(\overline{C}) = 0.$$

Next, we can establish the concavity of the value function.

Lemma 8. *P solving (D.9) on $[0, \overline{C}]$ subject to $P''(\overline{C}) = P'(\overline{C}) - 1 = 0$ is strictly concave of $[0, \overline{C})$. In addition, $P'''(C) > 0$ on $[0, \overline{C}]$.*

Proof. Take the ODE (D.9) and invoke the envelope theorem to differentiate the ODE wrt. C. This yields after rearranging (under the optimal controls):

$$P'''(C) = \frac{2}{(1-\beta)^2\sigma^2} \times \left(\frac{\delta P'(C)}{1-\lambda} - P''(C)\mu_C \right)$$

At the boundary we have that $P''(\overline{C}) = 0$ and $P'(\overline{C}) = 1$ so that $P'''(\overline{C}) > 0$. Hence, $P(C)''' > 0 > P''(C)$ in a left neighbourhood of $\overline{C}$.

Let us assume to the contrary that P is not strictly concave on $[0,\overline{C})$ and define $\tilde{C} \equiv \sup\{C \in [0,\overline{C}] : P''(C) > 0\}$. By assumption, the set over which we take the supremum is non-empty, so that $\tilde{C}$ is well defined with $\tilde{C} < \infty$. As $P''(C) < 0$ in a left-neighbourhood of $\overline{C}$, we also have that $\tilde{C} < \overline{C}$. Due to continuity, $P''(\tilde{C}) = 0$ so that $P'''(\tilde{C}) > 0$. Hence, there exists $\hat{C} > \tilde{C}$ with $v''(\hat{C}) > 0$, a contradiction to the definition of $\tilde{C}$. Hence, $P'' < 0$ on $[0,\overline{C})$.

If now $P'''(C) \leq 0$ for some C, then it must be that $\mu_C \leq 0$. However, due to $P''(C) < 0$, it follows — by means of (D.9) — that $P(C) < 0$, a contradiction. Thus, $P''' > 0$ on $[0,\overline{C}]$, which concludes the proof. ☐

Concavity implies that $P'(C) \geq 1$ on $[0,\overline{C}]$ with equality only at $\overline{C}$.

D.4.2 Part VI

In the beginning of the proof (i.e., in Part I), we have conjectured that the outside equity value P under the optimal payout agreement $\mathcal{C}$ solves (D.2) on the interior of the state space and that dividend payouts reflect back into the interior of the state space. Given this conjecture, we have shown that the state space can be reduced to $[0,\overline{C}]$ (in that C never leaves this interval), that the outside equity value solves(D.9) and that $dDiv = \max\{C - \overline{C}, 0\}$.

Lastly, we verify that P solving (D.9) subject to $P(\overline{C}) - 1 = P''(\overline{C}) = 0$ indeed represents the equity value under the optimal $\mathcal{C}$.

Lemma 9. *Assume that outside equity holders recover $L \geq 0$ upon liquidation at time τ. The outside equity value $P = P(C)$ under the optimal payout agreement $\mathcal{C}$ solves* (D.9). *The optimal payout agreement $\mathcal{C}$ stipulates $dDiv > 0$ if and only if $P'(C) = 1$ and payments w are characterized by (β, φ) determined through* (D.9).

Proof. We verify that $P(C_{t-})$ solving (D.9) indeed represents outside equity value in optimum, which proves the Lemma. To start with, note that given the proposed dividend policy $dDiv = \max\{C - \overline{C}, 0\}$ it follows that C never leaves $[0,\overline{C}]$. Because $P'(\overline{C}) = 1$ and P is strictly concave on $[0,\overline{C})$, we have $P'(C) \geq 1$ for $C \in [0,\overline{C}]$ with equality only at $\overline{C}$

Let $\mathcal{C} = (Div, w) \in \mathbf{C}$ the optimal payout agreement consider any other payout agreement $\hat{\mathcal{C}} = (\hat{D}iv, \hat{w}) \in \mathbf{C}$. Let us for brevity rewrite the SDE for C:

$$dC_t = \mu_{Ct}dt + \sigma_{tC}dZ_t - dDiv_t$$

with

$$\mu_{Ct} \equiv \left(r - \frac{\delta}{1-\varphi_t}\right) C_{t-} + \mu - \frac{\rho r}{2}(\beta_t \sigma)^2 \text{ and } \sigma_{tC} = (1-\beta_t)\sigma,$$

where we suppress the dependence of drift and volatility on controls and model parameters. Introduce the linear functional $\mathcal{L}$, operating on functions dependent on $C \geq 0$ with $\mathcal{L}f(C) = f'(C)\mu_C + \frac{\sigma_C^2 f''(C)}{2}$. Define for $t < \tau$ the auxiliary gain process upon following an arbitrary strategy induced by a payout agreement $\hat{\mathcal{C}}$ up to time t and then switching to the strategy induced by the payout agreement $\mathcal{C}$

$$G_t^P = G_t^P(\hat{\mathcal{C}}) = \int_0^t e^{-rs} d\hat{D}iv_s + e^{-rt}P(C_{t-}).$$

By Itô's Lemma:

$$\begin{aligned} e^{rt}dG_t^P &= \{-rP(C_{t-}) + \mathcal{L}P(C_{t-})\ \}\,dt + \big(1 - P'(C_{t-})\big)d\hat{D}iv + \sigma_{tC}P'(C_{t-})dZ_t \\ &\equiv \mu_t^G(\hat{\mathcal{C}})dt + \big(1 - P'(C_{t-})\big)d\hat{D}iv + \sigma_{tC}P'(C_{t-})dZ_t. \end{aligned}$$

By the HJB equation (D.9), the drift term in curly brackets, i.e., $\mu_t^G(\hat{\mathcal{C}})$, is zero under the optimal controls induced by the payout agreement $\mathcal{C}$, while each other strategy/payout agreement will make this term (weakly) negative, implying that $\mu_t^G(\hat{\mathcal{C}}, \hat{Div}) \leq 0$. Because the process $\hat{Div}$ is almost surely increasing and the fact that $P'(C_{t-}) \geq 1$, the term $\big(1 - P'(C_{t-})\big)$ is (weakly) negative under any dividend payout policy $\hat{Div}$ and zero under the payout policy Div.

Next, our regularity conditions ensure that β is bounded and so is σ_C. Further, P' and P must be bounded over $(0, \infty)$. Evidently, $P < \mu/r$. If now P' were not bounded, then P could not be bounded either; in addition, the assumption that P is twice continuously differentiable on the whole state space already implies a bounded first derivative. Hence, there exists $\infty > K > 0$ with $P, P' < K$. Hence:

$$\mathbb{E}\bigg(\int_0^t e^{-rs}\sigma_{tC}P'(C_{t-})dZ_s\bigg) = 0$$

for all $t < \tau$. Therefore, $G^P(\hat{\mathcal{C}})$ follows a supermartingale, while $G^P(\mathcal{C})$ follows a martingale under the measure $\mathbb{P}$ and so do the stopped processes $\{G^P(\hat{\mathcal{C}})_{t\wedge\tau}\}$ and $\{G^P(\mathcal{C})_{t\wedge\tau}\}$. Hence, the payoff $\hat{P}_0$ under any payout agreement $\hat{\mathcal{C}}$ satisfies

$$\hat{P}(C_{0-}) \equiv G_{0-}^P(\hat{\mathcal{C}}) \geq \mathbb{E}G_{t\wedge\tau^I}^P(\hat{\mathcal{C}})$$

Then it follows for any t:

$$\begin{aligned}
\hat{P}(C_{0-}) &= \mathbb{E}\bigg(\int_0^\tau e^{-rs}d\hat{Div}_s + e^{-r\tau}L\bigg) = \mathbb{E}G_\tau^P(\hat{\mathcal{C}}) + e^{-r\tau}L \\
&= \mathbb{E}\bigg(G_{t\wedge\tau^I}^P(\hat{\mathcal{C}}) + \mathbf{1}_{\{t\leq\tau\}}\Big[\int_t^\tau e^{-rs}d\hat{Div}_s + e^{-r\tau}L - e^{-rt}P(C_{t-})\Big]\bigg) \\
&= \mathbb{E}G_{t\wedge\tau}^P(\hat{\mathcal{C}}) + e^{-rt}\mathbb{E}_t\mathbf{1}_{\{t\leq\tau\}}\bigg(\int_t^\tau e^{-r(s-t)}d\hat{Div}_s + e^{-r(\tau-t)}L - P(C_{t-})\bigg) \\
&\leq P(C_{0-}) + e^{-rt}\big(\mu/r - L\big),
\end{aligned}$$

where we used the supermartingale property and the fact that

$$\mathbb{E}_t\bigg(\int_t^\tau e^{-r(s-t)}d\hat{Div}_s + e^{-r(\tau-t)}L\bigg) \leq \frac{\mu}{r}$$

and $P(C_{t-}) \geq L$.

From the above arguments, we readily obtain $\hat{P}(C_{0-}) \leq P(C_{0-})$ for any payout agreement $\hat{\mathcal{C}}$. On the other hand, under $\mathcal{C}$ the outside equity value $\hat{P}(C_{0-})$ achieves $P(C_{0-})$, as the above weak inequality holds in equality when $t \to \infty$. This concludes the proof. □

D.5 Part VII

The following Lemmata characterize the optimal liquidation policy τ, which concludes the proof.

Define $t_0^i = \inf\{t \geq t_0^{i-1} : M_{t-} = 0\}$ recursively as the i'th time cash-holdings M are equal to zero. Set $t_0^0 = 0$ and note that due to $C_0 > 0$ it must be that $\mathbb{P}(t_0^1 > 0) = 1$.

Lemma 10. *Assume that outside equity holders recover $L \geq 0$ upon liquidation at time τ. If and only if:*

$$P'(0)\left(\mu - \frac{\rho r\sigma^2}{2}\right) \leq L,$$

liquidation is optimal at time t_0^1 in that $\tau = t_0^1$ and $P(0) = L$. Otherwise, $\tau = \infty$.

Proof. Liquidation at time t_0^1 yields the assumed liquidation value L, in which case $P(0) = L$. Next, note that for $t = t_0^1$, we have that $M_t = Y_t = 0$. Thus, $\tau > t_0^1$ requires that $vol(dM_t) = vol(dY_t) = 0$ at $t = t_0^1$. This implies that $vol(dw_t) = \sigma$ and $vol(dY_t) = \beta_t\sigma - vol(dw_t) = 0$ so that $\beta_t = 1$. Thus, $\tau > t_0^1$ yields by (D.9) value:

$$P(0) = P'(0)\left(\mu - \frac{\rho r\sigma^2}{2}\right),$$

which was to show. It is clear that if liquidation is sub-optimal at time $t = t_0^1$, it is also sub-optimal at $t = t_0^i$ for $i > 1$, leading to $\tau = \infty$. □

Because $P'(0) > 0$, it follows that liquidation $\tau = t_0^1$ is optimal if:

$$\frac{2\mu}{r} \leq \rho\sigma^2.$$

Lemma 11. *If $\tau = t_0^1$, then $\mathbb{P}(\tau < \infty) = 1$.*

Proof. It suffices to show that $vol(dC) = \sigma(1 - \beta(C)) > 0$ for all $C \in [0, \overline{C}]$. By (D.9), optimal β is given by:

$$\beta(C) = \max\{\lambda, \beta^*(C)\} \text{ with } \beta^*(C) := \frac{-P''(C)}{\rho r P'(C) - P''(C)}.$$

It follows that $\beta(C) < 1$ as long as $|P''(C)|$ is bounded. Due to $P''' > 0$, it follows that $|P''(0)| < \infty \Rightarrow |P''(C)| < \infty$. Hence, if liquidation at $C = 0$ is optimal that $\beta(C) < 1$ for all $C \in [0, \overline{C}]$, implying that $\mathbb{P}(\tau < \infty) = 1$. □

Lemma 12. *If $\tau > t_0^1$, then $P''(C) \to -\infty$ as $C \to 0$.*

Proof. Suppose that at time $t = t_0^1$, we have that $C_t = 0$. Note that (by arguments presented in the proof of the previous Lemma) $\tau > t_0^1$ implies that $\beta(C) \to 1$ as $C \to 0$. The HJB-equation (D.9) implies that optimal β is characterized by:

$$\beta(C) = \max\{\lambda, \beta^*(C)\} \text{ with } \beta^*(C) := \frac{-P''(C)}{\rho r P'(C) - P''(C)}.$$

Because $P'(C) \geq 1$ for all $C \in [0, \overline{C}]$, the limit $\beta(C) \to 1$ requires $P''(C) \to -\infty$.

The proof is complete if we can show that there does not exist $\tilde{C} \in (0, \overline{C})$ with $P''(C) \to -\infty$ as $C \to \tilde{C}$. Assume to the contrary that there exists $\tilde{C} \in (0, \overline{C})$ with $P''(C) \to -\infty$ as $C \to \tilde{C}$. Then, it must be $P(C) \to \tilde{P}(\tilde{C})$ with $\tilde{P}$ solving the ODE:

$$r\tilde{P}(\tilde{C}) = \tilde{P}'(\tilde{C})\left(\left(r - \frac{\delta}{1-\lambda}\right)\tilde{C} - \frac{\rho r\sigma^2}{2} + \mu\right) \tag{D.12}$$

$\tilde{C}$ must satisfy the standard value matching, smooth pasting and super contact conditions:

$$\lim_{C\to\tilde{C}} P(C) = \tilde{P}(\tilde{C})$$
$$\lim_{C\to\tilde{C}} P'(C) = \tilde{P}'(\tilde{C})$$
$$\lim_{C\to\tilde{C}} P''(C) = \tilde{P}''(\tilde{C}).$$

Therefore: $\tilde{P}''(\tilde{C}) = -\infty$. However, note that differentiating (D.12) yields:

$$\frac{\delta \tilde{P}'(C)}{1-\lambda} = \tilde{P}''(C)\left(\left(r - \frac{\delta}{1-\lambda}\right)\tilde{C} - \frac{\rho r \sigma^2}{2} + \mu\right),$$

implying that $\tilde{P}'(\tilde{C}) = \pm\infty$. However, this implies — by means of (D.12) — that $\tilde{P}(\tilde{C}) = \pm\infty$. It then follows — by means of value matching — that $\lim_{C \to \tilde{C}} P(C) = \pm\infty$, a contradiction. □

Corollary 3. $\tau > t_0^1 \implies \mathbb{P}(\tau = \infty) = 1.$

Proof. This follows from the fact that $vol(dC) = \sigma(1-\beta(C)) \to 0$ as $C \to$ while $\mathbb{E}dC \to const > 0$ as $C \to 0$. □

E Additional Analytical Results

E.1 Proof of Corollary 1

All three claims are proven separately.

1. *Proof.* Differentiating the expression for β^* w.r.t. C yields

$$\partial_C \beta^*(C) = \frac{\partial \beta^*(C)}{\partial C} \propto -P'(C)P'''(C) + P''(C)P''(C).$$

Owing to $P', P''' > 0$ and $P''(\overline{C}) = 0$, there is $C' := \inf\{C \geq 0 : \partial_C \beta^*(C) < 0\} < \overline{C}$ and β^* strictly decreases in an open left neighbourhood of $\overline{C}$. As $\sigma \to 0$, clearly $\overline{C} \to 0$. By the super-contact condition, it follows that $P''(C) = o(\overline{C})$, while $P'(C)P'''(C) \neq o(\overline{C})$. Hence, $C' \uparrow 0$ as $\sigma \to 0$, which proves that for σ sufficiently low β^* decreases on $[0, \overline{C})$. □

2. *Proof.* Note that for $\beta^*(\overline{C}) = 0$ which immediately implies that there exists $\hat{C} = \inf\{C \in [0, \overline{C} : \beta^*(C) \geq \lambda\}$. It is obvious that $\hat{C} \leq \overline{C}$. Since $\beta^*(C) > 0$ for all $C < \overline{C}$ and $\overline{C} \to \overline{C}' > 0$ as $\lambda \to 0$, it follows that $\hat{C} \to 0$. Thus, for λ sufficiently small, it must be that $\hat{C} < \overline{C}$ and there exists exactly one value solving the equation $\beta(C) = \lambda$, which completes the proof. □

3. *Proof.* Let us postulate that wages w follow a continuous Ito process:

$$dw_t = \mu_{wt}dt + \sigma_{wt}dZ_t + \xi_{wt}dDiv_t.$$

We have to determine the drift μ_{wt} , volatility σ_{wt} and the dividend sensitivity ξ_{wt}, taking the optimal control $\varphi_t = \lambda \ \forall\ t \geq 0$ as given.

By definition: $Y_t = \frac{\varphi_t C_t}{1-\varphi_t}$. Using (11):

$$dW_t = \frac{\rho r}{2}(\beta_t \sigma)^2 dt + \beta_t(dX_t - \mu dt).$$

and (3):

$$dS_t = rS_t dt + dB_t + dw_t - c_t dt \quad \text{with} \quad c_t = rW_t,$$

it is straightforward calculate:

$$dY_t = rY_t dt + \frac{\rho r}{2}(\beta_t \sigma)^2 dt + \beta_t \sigma dZ_t - dw_t.$$

On the other hand, because $\varphi = \lambda$ is constant:

$$dY_t = \frac{\varphi_t dC_t}{1-\varphi_t} = \frac{\lambda dC_t}{1-\lambda}.$$

As $dDiv_t = \max\{C_t - \overline{C}, 0\}$, it follows that:

$$\xi_{wt} = \frac{\lambda}{1-\lambda}.$$

Next, we consider $dDiv_t = 0$. Taking (22) for $dDiv_t = 0$:

$$dC_t = \left(r - \frac{\delta}{1-\lambda}\right) C_t dt - \frac{\rho r}{2}(\beta_t \sigma)^2 dt + \mu dt + (1-\beta_t)\sigma dZ_t,$$

so that:

$$dw_t = r\frac{\lambda C_t}{1-\lambda}dt + \frac{\rho r}{2}(\beta_t\sigma)^2 dt + \beta_d Z_t - \frac{\lambda}{1-\lambda}dC_t.$$

we obtain after rearranging and collecting terms:

$$dw_t = \underbrace{\frac{1}{1-\lambda} \times \left[\frac{\rho r}{2}(\beta_t\sigma)^2 + \frac{\lambda\delta C_t}{1-\lambda} - \lambda\mu\right] dt}_{=\mu_{wt}} + \underbrace{\frac{\beta_t - \lambda}{1-\lambda}\sigma}_{=\sigma_{wt}} dZ_t.$$

□

E.2 Proof of Corollary ??

We prove all claims separately.

1. *Proof.* The formula for the stock-returns follows upon invoking Ito's Lemma (under the optimal controls):

$$\begin{aligned} dR_t &= \frac{dDiv_t + dP(C_t)}{P(C_t)} \\ &= \frac{dDiv_t + \mathbb{E}dP(C_t) + P'(C_t)(1-\beta(C_t))\sigma}{P(C_t)} \\ &= rdt + \underbrace{\frac{P'(C_t)(1-\beta(C_t))\sigma}{P(C_t)}}_{=:\Sigma(C_t)} \end{aligned}$$

where we used the HJB-equation under the optimal controls:

$$rP(C_t)dt = dDiv_t + \mathbb{E}dP(C_t).$$

□

2. *Proof.* Recall that

$$\beta(C) = \max\{\lambda, \beta^*(C)\} \text{ with } \beta^*(C) := \frac{-P''(C)}{\rho r P'(C) - P''(C)}.$$

We can rewrite for $\beta(C) > \lambda$:

$$\Sigma(C) = (1-\beta)\sigma\frac{P'(C)}{P(C)} = \sigma\rho r \times \frac{(P'(C))^2}{P(C)\big(\rho r P'(C) - P''(C)\big)},$$

in which case

$$\begin{aligned}\Sigma'(C) &\propto 2P(C)\big(\rho r P'(C) - P''(C)\big)P'(C)P''(C) \\ &- (P'(C))^2 \times \Big[P'(C)\big(\rho r P'(C) - P''(C)\big) + P(C)\big(\rho r P''(C) - P'''(C)\big)\Big] \\ &= -(P'(C))^3 \times \big(\rho r P'(C) - P''(C)\big) + o\big(P(C)\big)\end{aligned}$$

If liquidation is optimal, then $P(0) = 0$ so that $\Sigma'(C) > 0$ for C sufficiently small. If liquidation is not optimal, then $\beta(C) \to 1$ and $\Sigma(C) \to 0$ as $C \to 0$ which immediately implies that $\Sigma'(C) > 0$ for C sufficiently small.

Next, note that in a neighbourhood of $\overline{C}$, we have that $\beta(C) = \lambda$, provided $\lambda > 0$, in which case it is clear that

$$\Sigma(C) = (1-\lambda)\sigma\frac{P'(C)}{P(C)}$$

decreases in this neighbourhood of $\overline{C}$ (as $P'(C) > 0 > P''(C)$). Otherwise, if $\lambda = 0$, the claim follows by continuity. □

3. *Proof.* Follows by continuity after taking the limit $\lambda \to 1$ (which leads to $\overline{C} \to 0$). □

E.3 Proof of Corollary 2

We first establish auxiliary results and then prove all claims of the Corollary separately.

E.3.1 Auxiliarly Results

Here, η is an arbitrary model parameter and define $\partial_\eta(\cdot) \equiv \frac{\partial(\cdot)}{\partial\eta}$.

We state the following auxiliary lemma:

Lemma 13. *For $\tau = \inf\{t \geq 0 : M_t = 0\}$ and $dC_t = \mu_{Ct}dt + \sigma_{Ct}dZ_t$ the following holds:*

$$\frac{dP(C)}{d\eta} = \frac{\partial P(C)}{\partial\eta} = \mathbb{E}\left[\int_0^\tau e^{-rt}\left(P'(C_t)\partial_\eta\mu_{Ct} + \frac{P''(C_t)}{2}\partial_\eta\sigma_{Ct}^2 - P(C_t)\partial_\eta r\right)dt\middle|C_0 = C\right].$$

Proof. Let η a model parameter and β, φ the optimal controls in optimum. Let $C \in [0, \overline{C}]$ and take the derivative

$$\frac{dP(C)}{d\eta} = \partial_\eta P(C) + \partial_{\overline{C}}P(C) \times \partial_\eta\overline{C} = \partial_\eta P(C),$$

where $\partial_{\overline{C}}P(C) = 0$ by means of the envelope theorem.

Let $P_\eta(C) = \partial_\eta P(C)$ and $P_{\eta\eta}(C) = \partial_{\eta\eta}P(C)$. Accordingly, differentiating the HJB equation (under the optimal controls) for any $C \in [0, \overline{C}]$:

$$rP(C) = \mu_C P'(C) + \frac{P''(C)}{2}\sigma_C^2$$

w.r.t. η yields:

$$P(C)\partial_\eta r = -\,rP_\eta(C) + P'(C)\partial_\eta \mu_C + \frac{P''(C)}{2}\partial_\eta \sigma_C^2 + \mu_C P_\eta'(C) + \sigma_C^2 P_\eta''(C)$$

where $\partial_\beta P(C) = \partial_\varphi P(C) = 0$ by the envelope theorem. The boundary conditions are $P_\eta'(\overline{C}) = P_\eta''(\overline{C}) = 0$. Note that provided smoothnesss, we can interchange the order of differentiation, such that:

$$P_\eta'(C) \equiv \frac{\partial}{\partial \eta}\frac{\partial P(C)}{\partial C} = \frac{\partial}{\partial C}\frac{\partial P(C)}{\partial \eta} \text{ and } P_\eta''(C) \equiv \frac{\partial}{\partial \eta}\frac{\partial^2 P(C)}{\partial C^2} = \frac{\partial^2}{\partial C^2}\frac{\partial P(C)}{\partial \eta}.$$

We can solve for:

$$rP_\eta(C) = \left(P'(C)\partial_\eta \mu_C + \frac{P''(C)}{2}\partial_\eta \sigma_C^2 - P(C)\partial_\eta r\right) + \frac{\mathbb{E}dP_\eta(C)}{dt}.$$

Invoking the Feynman-Kac formula yields the desired integral expression for $P_\eta(C)$. □

Lemma 14.

$$\frac{d\overline{C}}{d\eta} = \frac{1-\lambda}{\delta}\left(\partial_\eta \mu_C(\overline{C}) - rP_\eta(\overline{C}) - P(\overline{C})\partial_\eta r\right).$$

Proof. We evaluate the HJB-equation (23) at the boundary $C = \overline{C}$:

$$rP(\overline{C}) = \left(r - \frac{\delta}{1-\lambda}\right)\overline{C} - \frac{\rho r}{2}(\lambda\sigma)^2 + \mu. \tag{E.1}$$

Differentiate this identity wrt. η:

$$P(\overline{C})\partial_\eta r + rP_\eta(\overline{C}) + rP'(\overline{C})\frac{d\overline{C}}{d\eta} = \left(r - \frac{\delta}{1-\lambda}\right)\frac{d\overline{C}}{d\eta} + \partial_\eta \mu_C(\overline{C}).$$

Note that $P'(\overline{C}) = 1$ and rearrange:

$$\begin{aligned}\frac{d\overline{C}}{d\eta} &= \frac{1-\lambda}{\delta}\left(\partial_\eta \mu_C(\overline{C}) - rP_\eta(\overline{C}) - P(\overline{C})\partial_\eta r\right) \\ &\propto \partial_\eta \mu_C(\overline{C}) - rP_\eta(\overline{C}) - P(\overline{C})\partial_\eta r.\end{aligned}$$

□

E.3.2 Claim 1

Proof. Note that by Lemma 13:

$$P_\sigma(C) = \partial_\sigma P(C) = \mathbb{E}\left[\int_0^\tau e^{-rt}\Big(-\rho r\beta_t^2\sigma P'(C_t) + \sigma(1-\beta_t)^2 P''(C_t)\Big)dt \,\Big|\, C_0 = C\right] < 0.$$

Next, note that:

$$\partial_\sigma \mu_C(\overline{C}) = -\rho r\lambda^2\sigma.$$

Lemma 14 now implies that $\overline{C}$ increases in σ, provided λ or ρ or σ is sufficiently small.

Note that:

$$\lim_{\lambda \to 1} \overline{C} = 1,$$

as the HJB (23) would otherwise imply — due to $P'(C) \geq 1$ — a value $P(C) < 0$ for any $C \in [0, \overline{C}]$. Recall that by Proposition 2:

$$\tau = \infty \text{ a.s.} \iff 2\mu \geq \rho r \sigma^2.$$

It follows that:

$$\lambda = 1 \;\wedge\; 2\mu < \rho r \sigma^2 \implies \tau = 0 \text{ a.s.} \implies P_\sigma(C) = 0.$$

Thus, if $\rho r \sigma^2 > 2\mu$, then:

$$P_\sigma(C) \to 0 \text{ as } \lambda \to 1.$$

uniformly for all $C \in [0, \overline{C}]$. Thus, if $\rho r \sigma^2 > 2\mu$, we always find $\lambda \in (0, 1)$, such that

$$\frac{d\overline{C}}{d\sigma} \propto -\rho r \lambda^2 - P_\sigma < 0,$$

which was to show. □

E.3.3 Claim 2

Proof. Note that by Lemma 13:

$$P_\lambda(C) = \mathbb{E}\left[\int_0^\tau e^{-rt}\left[\left(-\rho r \lambda \sigma^2 P'(C_t) - \sigma^2(1-\lambda)P''(C_t)\right)\mathbf{1}_{\{\beta_t = \lambda\}} - \frac{P'(C_t)C_t\delta}{(1-\lambda)^2}\right] dt \middle| C_0 = C\right].$$

Note that:

$$-\rho r \lambda \sigma^2 P'(C_t) - \sigma^2(1-\lambda)P''(C_t) > 0 \Longrightarrow \beta_t > \lambda,$$

so that:

$$P_\lambda(C) < 0.$$

Clearly:

$$\partial_\lambda \mu_C(C) = -\frac{\delta \overline{C}}{(1-\lambda)^2} - \rho r \lambda \sigma^2 < 0,$$

so that by virtue of Lemma 14:

$$\frac{d\overline{C}}{d\lambda} < 0.$$

□

E.3.4 Claim 3

Proof. Look at the target cash-holdings $\overline{M} = \overline{C}/(1-\lambda)$ and differentiate wrt. λ:

$$\begin{aligned}
\frac{d\overline{M}}{d\lambda} &= \frac{\overline{C}}{(1-\lambda)^2} + \frac{1}{1-\lambda}\frac{d\overline{C}}{d\lambda} \\
&\propto \frac{\overline{C}}{1-\lambda} + \frac{d\overline{C}}{d\lambda} \\
&= \frac{\overline{C}}{1-\lambda} + \frac{1-\lambda}{\delta}\left(\partial_\lambda \mu_C(\overline{C}) - rP_\lambda(\overline{C})\right) \\
&= \frac{\overline{C}}{1-\lambda} + \frac{1-\lambda}{\delta}\left(-\frac{\delta\overline{C}}{(1-\lambda)^2} - \rho r\lambda\sigma^2 - rP_\lambda(\overline{C})\right) \\
&= -\frac{(1-\lambda)(rP_\lambda(\overline{C}) + \rho r\lambda\sigma^2)(\overline{C})}{\delta} \\
&\propto -P_\lambda(\overline{C}) - \rho r\lambda\sigma^2.
\end{aligned}$$

Recall that by Lemma 13:

$$P_\lambda(C) = \mathbb{E}\left[\int_0^\tau e^{-rt}\left[\left(-\rho r\lambda\sigma^2 P'(C_t) - \sigma^2(1-\lambda)P''(C_t)\right)\mathbf{1}_{\{\beta_t=\lambda\}} - \frac{P'(C_t)C_t\delta}{(1-\lambda)^2}\right]dt\,\middle|\, C_0 = C\right],$$

so that $P_\lambda < 0$ also in the limit $\lambda \to 0$ or $\rho \to 0$. Therefore, for $\lambda \to 0$, it follows that $d\overline{M}/d\lambda > 0$ implying that $d\overline{M}/d\lambda > 0$ for λ or ρ sufficiently small.

On the other hand, it follows that if $2\mu < \rho r\sigma^2$:

$$\lim_{\lambda\to 1}\tau = 0 \;\text{ a.s.} \quad \Longrightarrow \quad \lim_{\lambda\to 1} P_\lambda(\overline{C}) = 0.$$

Hence, by continuity, for ρ and λ sufficiently large: $d\overline{M}/d\lambda < 0$. □

F Proof of Lemma XXX

We will now show that this is However, we assume that the insider cannot commit to inject funds during a refinancing event and may falsely report the amount that she contribute. Crucially, new outside investors do not observe how much cash the insider contributes during refinancing or when the insider fails to inject any cash at all, implying that outsiders' contributions Δ_t^M cannot be contingent on the insider's contributions. As a result, refinancing is subject to moral hazard. Therefore, the rewards α and γ are set to provide sufficient incentives to the insider.

To make the argument in the preceding paragraph more precise, consider the insider's available actions when the firm has cash M_t and future promised payouts to the insider worth Y_t (i.e., $C_t = M_t - Y_t$). When a refinancing opportunity arises over $[t, t+dt]$, that is, $d\Pi_t = 1$, the insider may abscond with the total cash-balance $M_t + \Delta_t^M$ and renege to inject funds, just after outside investors contribute amount Δ_t^M.

Absconding yields i) a bonus payment, if $\alpha_t > 0$, and ii) diverted funds $\lambda M_{t+dt} = \lambda(M_t + \Delta_t)$. Because the insider optimally reneges to inject cash, in case she absconds, absconding yields in total payoff $\lambda(M_t + \Delta_t) + \max\{\alpha_t, 0\}$, whereas complying to the payout agreement yields payoff

$Y_{t+dt} = Y_t + \gamma_t + \alpha_t$. This leads to the incentive condition

$$\lambda(M_t + \Delta_t) + \max\{\alpha_t, 0\} \leq Y_t + \alpha_t + \gamma_t. \tag{F.1}$$

Notably, (33) reveals that bonus payments $\alpha_t > 0$ do not generate incentives in that α_t cancels from the incentive condition, if positive. By contrast, $\alpha_t < 0$ even tightens incentive compatibility (F.1) and so generates dis-incentives. Because the risk-averse insider is in addition financially more "constrained" than the new risk-neutral outside investors, it is sub-optimal to ask her to contribute funds during the refinancing event, and thus $\alpha_t \geq 0$.

Hence, the appropriate incentives during refinancing require promising the insider a sufficiently large deferred compensation package $Y_t + \gamma_t$ just after refinancing, implying that Y_t (or equivalently φ_t) or γ_t must be large. Since the incentive constraint (33) tightens when more funds Δ_t are raised, the firm might decide to raise less funds due to agency conflicts. Thus, at the optimum, (33) holds with equality and $\alpha_t \geq 0$, so as to minimize the agency costs associated with the payout agreement.

G Sketch of the Proof of Proposition 3

Proposition 3 can be proven exactly the same way as Proposition 2. We mainly present here heuristic arguments that lead to the key equations. A standard verification proof can be applied to verify the correctness of the heuristic arguments.

G.1 Part I

Proof. Here, we derive the law of motion for the continuation utility (8). Like in Proposition 1, the martingale representation theorem can be applied to derive a stochastic differential equation for U.

Specifically, by the martingale representation theorem, there exist processes β, Γ such that:

$$dU_t = rU_t dt - u(c_t)dt + \beta_t(-\rho r U_t)(dX_t - \mu dt) + \Gamma_t(-\rho r U_t)(d\Pi_t - \pi dt).$$

First, note that — due to consumption smoothing — $rU_t = u(c_t)$ so that

$$dU_t = \beta_t(-\rho r U_t)(dX_t - \mu dt) + \Gamma_t(-\rho r U_t)(d\Pi_t - \pi dt).$$

Take the continuation value in monetary units (certainty equivalent)

$$W(U) := \frac{-\ln(-\rho r U)}{\rho r} \tag{G.1}$$

and apply Ito's Formula to obtain:

$$dW = \frac{\rho r}{2}(\beta\sigma)^2 dt + \beta\sigma dZ - \frac{\ln(1-\rho r\Gamma)}{\rho r} d\Pi - \pi\Gamma dt.$$

Define:

$$\hat{\Gamma} := -\frac{\ln(1-\rho r\Gamma)}{\rho r} \iff \Gamma = \frac{(1-e^{-\rho r\hat{\Gamma}})}{\rho r},$$

leading to:

$$dW = \frac{\rho r}{2}(\beta\sigma)^2 dt + \beta\sigma dZ + \hat{\Gamma} d\Pi - \frac{\pi(1-e^{-\rho r\hat{\Gamma}})}{\rho r} dt.$$

Next, we decompose

$$\hat{\Gamma} =: \alpha + \gamma,$$

which yields (34).

Next, we can calculate

$$dY = dW - dS = rY dt - dw + \frac{\rho r}{2}(\beta\sigma)^2 dt + \beta\sigma dZ + \hat{\Gamma} d\Pi - \frac{\pi(1 - e^{-\rho r\hat{\Gamma}}}{\rho r} dt.$$

By assumption, cash holdings evolve according to:

$$dM = (r - \delta)M dt + \mu dt + \sigma dZ + \Delta^M d\Pi - dDiv - dw,$$

where we define $\Delta := \Delta^M - \alpha$. As a result, direct calculations yield:

$$dC = dM - dY = \mu_C dt + (1 - \beta)dZ + (\Delta - \gamma)d\Pi - dDiv$$

with

$$\mu_C := \left(r - \frac{\delta}{1 - \varphi}\right) C + \mu - \frac{\rho r}{2}(\beta\sigma)^2 + \frac{\pi(1 - e^{-\rho r(\alpha+\gamma)})}{\rho r}, \tag{G.2}$$

which is (35). □

G.2 Part II

Let us heuristically derive the HJB equation (37). The fact that the outside equity value under the optimal payout agreement indeed solves (37) can be verified using the arguments presented in the proof of Proposition 2

Proof. As shown in the proof of Proposition 2, we conjecture that C is a sufficient statistic for the optimization problem. We also conjecture that dividends take the form $dDiv = \max\{C - \overline{C}, 0\}$ where $\overline{C}$ is the endogeneous payout boundary, determined by the standard smooth pasting and super contact conditions:

$$P'(\overline{C}) - 1 = P''(\overline{C}) = 0.$$

Under these conjectures, by the dynamic programming principle, the outside equity value must solve the HJB equation on $[0, \overline{C}]$:

$$rP(C)dt = \max_{\beta \geq \lambda, \varphi \geq \lambda, \alpha, \gamma, \Delta} \mathbb{E} dP(C).$$

Under the conjecture that P only depends on C, we can expand by means of Ito's Lemma, yielding:

$$rP(C) = \max_{\beta \geq \lambda, \varphi \geq \lambda, \alpha, \gamma, \Delta} \left\{ P'(C)\mu_C + \pi\big[P(C + \Delta - \gamma) - P(C) - \Delta - \alpha\big] + P''(C)\frac{\sigma^2(1 - \beta)^2}{2} \right\},$$

which is equation (37), as desired. If the controls only depend on C, the right hand side only depends on C for any φ. In this case, $P(C)$ only depends on C and so do the derivative $P'(C), P''(C)$. Then, the controls all depend only on C and so does then the left hand side, i.e., $P(C)$. That is, P only depends on C, thereby verifying our conjecture that C is the only payoff relevant state. □

G.3 Part III

Let us characterize the termination policy τ.

Lemma 15. *Liquidation at the first time cash holdings fall down to zero is optimal if and only if:*

$$P'(0)\left(\mu - \frac{\rho r\sigma^2}{2} + \frac{\pi(1-e^{-\rho r(\alpha(0)+\gamma(0))})}{\rho r}\right) + \pi(P(\Delta(0)-\gamma(0)) - \Delta(0) - \alpha(0) \leq 0. \tag{G.3}$$

In this case, $\mathbb{P}(\tau < \infty) = 1$*. Otherwise, if* (G.3) *does not hold,* $\tau = \infty$ *almost surely. Moreover:*

$$2\mu \geq \rho r\sigma^2 \implies \tau = \infty \quad a.s.\ .$$

Proof. By (37), it follows in all state $C > 0$ that $Y = \lambda M$. As $C_0 > 0$ (see proof of Proposition 2), it follows that

$$M = Y = 0 \iff C = 0.$$

To preclude liquidation at $C = 0$, it must be that:

$$vol(dY) = vol(dM) = 0 \implies \beta = 1,$$

as neither M nor Y can fall below zero. By the HJB equation (37), this yields payoff

$$P'(0)\left(\mu - \frac{\rho r\sigma^2}{2} + \frac{\pi(1-e^{-\rho r(\alpha(0)+\gamma(0))})}{\rho r}\right) + \pi(P(\Delta(0)-\gamma(0)) - \Delta(0) - \alpha(0),$$

while liquidation yields payoff zero. Thus, $\mathbb{P}(\tau = \infty) = 1$ is optimal if and only if (G.3) is *not* met.

Last, note that because under the optimal controls

$$\pi(P(\Delta(0)-\gamma(0)) - \Delta(0) - \alpha(0) \geq 0,$$

the parameter condition $2\mu \geq \rho r\sigma^2$ readily implies that (G.3) is not met, thereby concluding the proof. □

H Additional Results with Refinancing

Define:

$$\begin{aligned}
\mu_{Ct^-} &:= \left(r - \frac{\delta}{1-\varphi_t}\right)C_{t^-} + \mu - \frac{\rho r}{2}(\beta_t\sigma)^2 + \frac{\pi(1-e^{-\rho r(\alpha_t+\gamma_t)})}{\rho r} \\
\pi_{Ct^-} &:= \pi(P(C_{t^-} + \Delta_t - \gamma_t) - P(C_{t^-}) - \Delta_t - \alpha_t).
\end{aligned}$$

Taking the HJB equation (37), it is straightforward to show — mimicking the proof of Lemma 13 — that:

Lemma 16. *For* $\tau = \inf\{t \geq 0 : M_t = 0\}$ *and* $dC_t = \mu_{Ct^-}dt + \sigma_{Ct^-}dZ_t + J_{t^-}d\Pi_t$*:*

$$\frac{dP(C)}{d\eta} = \mathbb{E}\left[\int_0^\tau e^{-rt}\left(P'(C_{t^-})\partial_\eta\mu_{Ct^-} + \frac{P''(C_{t^-})}{2}\partial_\eta\sigma^2_{Ct^-} + \partial_\eta\pi_{Ct^-} - P(C_{t^-})\partial_\eta r\right)dt\middle|C_0 = C\right].$$

Proof. The proof is identical to the proof of Lemma 13 and therefore omitted. □

H.1 Proof of Corollary ??

Proof. It follows from the main text that $C^*(C) = \overline{C}$ and that $\gamma(C) = 0$. The optimal α solves $\frac{\partial P(C)}{\partial \alpha} = 0$ and it is immediate to derive (**??**). Claim 1 is immediate because of the concavity of the value function.

To prove claim 2, it suffices to show that:

$$\frac{dP'(C)}{d\sigma} < 0$$

for any fixed $C \in (0, \overline{C})$. By Lemma 16:

$$\frac{dP(C)}{d\sigma} = \mathbb{E}\left[\int_0^\tau e^{-rt}\Big(- \rho r \beta_t^2 \sigma P'(C_{t-}) + \sigma(1-\beta_t)^2 P''(C_{t-})\Big) dt \middle| C_0 = C\right]$$

Because of $2\mu \geq \rho r \sigma^2$, it follows by Lemma 15 that $\tau = \infty$.

Take the integrand:

$$-\rho r \beta_t^2 \sigma P'(C_{t-}) + \sigma(1-\beta_t)^2 P''(C_{t-}) \propto -\rho r \beta_t^2 P'(C_{t-}) + (1-\beta_t)^2 P''(C_{t-}) =: A(C_{t-}, \beta_t)$$

Note that — by virtue of (37) — in any state $C = C_{t-}$:

$$\beta_t = \arg\min_{\beta \geq \lambda} A(C, \beta).$$

Fix any $\beta = \bar{\beta} \in [0, 1]$ and note that:

$$\partial_C A(C, \bar{\beta}) = -\rho r \bar{\beta}^2 P''(C) + (1-\bar{\beta})^2 P'''(C) > 0,$$

in that for all $\bar{\beta} \in [0, 1]$, $A(C, \bar{\beta})$ increases (in the interior of the state space $(0, \overline{C})$). Thus, for any $C_1, C_2 \in (0, \overline{C})$ with $C_1 < C_2$:

$$A(C_1, \bar{\beta}) < A(C_2, \bar{\beta}) \implies \min_{\beta \geq \lambda} A(C_1, \beta) < A(C_2, \bar{\beta}).$$

As this holds for all $\bar{\beta}$, it must be that:

$$\min_{\beta \geq \lambda} A(C_1, \beta) < \min_{\beta \geq \lambda} A(C_2, \beta),$$

implying that $A(C, \beta)$ increases in C. Because $\tau = \infty$, $dP(C)/d\sigma$ increases in C if its integrand $A(C, \beta)$ increases in C. As a result:

$$\frac{\partial}{\partial C} \frac{dP(C)}{d\sigma} = \frac{dP'(C)}{d\sigma} > 0,$$

where we have interchanged the order of differentiation as $P \in C^2$ is smooth in state and parameter variables. $\square$

H.2 Proof of Corollary

The optimal refinancing policy is determined after going through the optimization implied by the HJB equation (37). The steps leading to the results are already presented in the main text and so not repeated here.

We prove all remaining claims (i.e., claims 1,2,3) separately.

1. *Proof.* We partition the state space such that $[0,\overline{C}] = \mathcal{S}_1 \cup \mathcal{S}_2$ with $\mathcal{S}_1 := \{C \in [0,\overline{C}] : \alpha(C) = 0\}$ and $\mathcal{S}_2 := \{C \in [0,\overline{C}] : \alpha(C) > 0$.

   First, consider $\alpha(C) > 0$, i.e., consider $C \in \mathcal{S}_2$. In this case $\overline{C} = C^*(C)$ and

$$\gamma = \frac{\lambda(\overline{C} - C)}{1-\lambda} \quad \text{and } \Delta = \frac{\overline{C} - C}{1-\lambda}$$

   decrease in C. In addition,

$$\alpha(C) = \frac{\ln(P'(C))}{\rho r} - \frac{\lambda(\overline{C} - C)}{1-\lambda},$$

   so that:

$$\Delta^M(C) = \Delta(C) + \alpha(C) = \frac{\ln(P'(C))}{\rho r} + \overline{C} - C$$

   also decreases in C, as $P''(C) \leq 0$.

   Second, consider $\alpha(C) = 0$, i.e., consider $C \in \mathcal{S}_1$. In this case, $\Delta^M(C) = \Delta(C)$ and $\gamma(C) = \lambda\Delta(C)$. Assume to the contrary that there exist $C_1, C_2 \in \mathcal{S}_1$ with $C_1 < C_2$ and $\Delta(C_1) \leq \Delta(C_2)$. This clearly implies $C^*(C_2) > C^*(C_1)$, so that $P'(C^*(C_2)) < P'(C^*(C_1))$ by concavity. Likewise: $P'(C_2) < P'(C_1)$. As $C_i \in \mathcal{S}_1$, it must be that:

$$P'(C^*(C_i)) = 1 + \frac{\lambda}{1-\lambda}\left[1 - P'(C_i)e^{-\rho r\lambda\Delta(C_i)}\right] \quad \text{for} \quad i \in \{1,2\},$$

   which implies (after subtracting the two equalities):

$$P'(C^*(C_1)) - P'(C(C_2)) = \frac{\lambda}{1-\lambda}\left(P'(C_2)e^{-\rho r\lambda\Delta(C_2)} - P'(C_1)e^{-\rho r\lambda\Delta(C_1)}\right)$$

   The left-hand side is bigger than zero. The right hand has the same sign as:

$$P'(C_2)e^{-\rho r\Delta(C_2)} - P'(C_1)e^{-\rho r\Delta(C_1)} \leq e^{-\rho r\Delta(C_2)}(P'(C_2) - P'(C_1)) < 0,$$

   where the first inequality uses $\Delta(C_2) \geq \Delta(C_1)$ and the second one $P'(C_2) < P'(C_1)$. Thus, we have achieved the desired contradiction, which concludes the proof.

   □

2. *Proof.* Because of $C^*(C) \leq \overline{C}$, it follows that $C^*(\overline{C}) = 0$ and $C^*(C) > 0$ for $C < \overline{C}$. The claim follows by continuity. □

3. *Proof.* If $\alpha(C) > 0$, i.e., $C \in \mathcal{S}_2$, then $\overline{C} = C^*(C)$ and the claim follows.

   If $\alpha(C) = 0$, i.e., $C \in \mathcal{S}_1$, then $C^*(C)$ solves

$$P'(C^*(C)) = 1 + \frac{\lambda}{1-\lambda}\left[1 - P'(C)e^{-\rho r\lambda\Delta(C)}\right] \quad \text{with} \quad \Delta(C) = \frac{C^*(C) - C}{1-\lambda}$$

We implicitly differentiate wrt. C and obtain:

$$P''(C^*)\frac{\partial C^*}{\partial C} = \frac{\lambda e^{-\rho r \frac{\lambda}{1-\lambda}[C^*-C]}}{1-\lambda}\left[-P''(c) + \frac{\rho r \lambda}{1-\lambda}P'(C)\left(\frac{\partial C^*}{\partial C} - 1\right)\right],$$

so that:

$$\frac{\partial C^*}{\partial C}\left(P''(C^*) - \frac{\rho r \lambda}{1-\lambda}P'(C)\frac{\lambda e^{-\rho r \frac{\lambda}{1-\lambda}[C^*-C]}}{1-\lambda}\right) = \frac{\lambda e^{-\rho r \frac{\lambda}{1-\lambda}[C^*-C]}}{1-\lambda}\left(-P''(C) - \frac{\rho r \lambda}{1-\lambda}P'(C)\right).$$

Next, note that owing to $P''(C) < 0 < P'(C)\ \forall\ C < \overline{C}$:

$$\begin{aligned} \frac{\lambda e^{-\rho r \frac{\lambda}{1-\lambda}[C^*-C]}}{1-\lambda} &> 0 \\ P''(C^*) - \frac{\rho r \lambda}{1-\lambda}P'(C)\frac{\lambda e^{-\rho r \frac{\lambda}{1-\lambda}[C^*-C]}}{1-\lambda} &< 0. \end{aligned}$$

As a result, $\frac{\partial C^*}{\partial C}$ has the same sign as:

$$P''(C) + \frac{\rho r \lambda}{1-\lambda}P'(C) = P''(C) + o(\rho\lambda).$$

Note that $P''(C) < 0$ with $P'''(C) > 0$. Thus, if ρ or λ sufficiently small and $C < \overline{C}$, then:

$$\frac{\partial C^*}{\partial C} < 0,$$

which was to show. □

4. *Proof.* Invoke Lemma 16 to write:

$$\frac{dP(C)}{d\lambda} = \mathbb{E}\Big[\int_0^\tau e^{-rt}\left(\mathcal{X}_1(C_t)\mathbf{1}_{\{\beta_t=\lambda\}} + \mathcal{X}_2(C_t)\mathbf{1}_{\{\alpha_t=0\}}\right)dt\Big|C_0 = C\Big] < 0,$$

for some endogeneous bounded, continuous, negative functions $\mathcal{X}_i(\cdot)$ with $i \in \{1,2\}$. As $\beta(C) > 0$ for all $\lambda \geq 0$ for $C < \overline{C}$, it follows that $\mathcal{X}_1(C_t)\mathbf{1}_{\{\beta_t=\lambda\}} \to 0$ for all $t \geq 0$ almost surely as $\lambda \to 0$. Likewise, as $\alpha(C) > 0$ for all $C \in [0,\overline{C})$ if $\lambda \to 0$,it follows that $\mathcal{X}_2(C_t)\mathbf{1}_{\{\alpha_t=0\}} \to 0$ for all $t \geq 0$ almost surely as $\lambda \to 0$. As a result, $dP(C)/d\lambda \to 0$ as $\lambda \to 0$. As this holds for any C, it follows that $\partial_C dP(C)/d\lambda = dP'(C)/d\lambda \to 0$ as $\lambda \to 0$ (where we have exploited the sufficient smoothness to interchange the order of differentiation). Hence, $dP'(C)/d\lambda = o(\lambda)$.

Next, consider $\alpha(C) = 0$, i.e., $C \in \mathcal{S}_1$, in which case the claim regarding α follows and $C^*(C)$ solves:

$$P'(C^*(C)) = 1 + \frac{\lambda}{1-\lambda}\left[1 - P'(C)e^{-\rho r \lambda \Delta(C)}\right] \quad \text{with} \quad \Delta(C) = \frac{C^*(C) - C}{1-\lambda}.$$

We differentiate this identity wrt. λ and obtain:

$$\begin{aligned}
&P''(C^*(C))\frac{dC^*(C)}{d\lambda}\\
=&\frac{1}{(1-\lambda)^2}\left[1-P'(C)e^{-\rho r\lambda\Delta(C)}\right]+\frac{\rho r\lambda P'(C)}{1-\lambda}\frac{d(\lambda\Delta(C))}{d\lambda}-\frac{\lambda e^{-\rho r\lambda\Delta(C)}}{1-\lambda}\frac{dP'(C)}{d\lambda}.
\end{aligned}$$

Calculate:

$$\frac{d(\lambda\Delta(C))}{d\lambda}=\frac{C^*(C)-C}{(1-\lambda)^2}+\frac{\lambda}{1-\lambda}\frac{dC^*(C)}{d\lambda}.$$

Hence, we obtain:

$$\begin{aligned}
&\left(P''(C^*(C))-\frac{\lambda}{1-\lambda}\frac{\rho r\lambda P'(C)}{1-\lambda}\right)\frac{dC^*(C)}{d\lambda}\\
=&\frac{1}{(1-\lambda)^2}\left[1-P'(C)e^{-\rho r\lambda\Delta(C)}\right]+\frac{\rho r\lambda P'(C)}{1-\lambda}\frac{C^*(C)-C}{(1-\lambda)^2}-\frac{\lambda e^{-\rho r\lambda\Delta(C)}}{1-\lambda}\frac{dP'(C)}{d\lambda}.
\end{aligned}\tag{H.1}$$

Owing to $C\in\mathcal{S}_1$ and $\alpha(C)=0$, it follows that $\left[1-P'(C)e^{-\rho r\lambda\Delta(C)}\right]>0$. For C sufficiently small, we have shown that $dP'(C)/d\lambda=o(\lambda)$ so that $-\frac{\lambda e^{-\rho r\lambda\Delta(C)}}{1-\lambda}\frac{dP'(C)}{d\lambda}=o(\lambda)$. Thus, the RHS of (H.1) is positive for λ sufficiently small. On the other hand, due to concavity:

$$P''(C^*(C))-\frac{\lambda}{1-\lambda}\frac{\rho r\lambda P'(C)}{1-\lambda}<0,$$

implying that

$$\frac{dC^*}{d\lambda}<0\ \text{ for }\ \lambda\ \text{ sufficiently small.}$$

Consider that $\alpha(C)>0$, i.e., $C\in\mathcal{S}_2$. Under these circumstances, $C^*(C)=\overline{C}$ so that the claim regarding $C^*(C)$ follows. In addition:

$$\alpha(C)=\frac{\ln(P'(C))}{\rho r}-\frac{\lambda(\overline{C}-C)}{1-\lambda},$$

so that:

$$\frac{d\alpha(C)}{d\lambda}=-\frac{\overline{C}-C}{(1-\lambda)^2}+\frac{1}{\rho rP'(C)}\frac{dP'(C)}{d\lambda}$$

which is strictly negative for λ sufficiently small, because of $dP'(C)/d\lambda=o(\lambda)$. □

I Supplementary Appendix

I.1 Endogeneous Carry Cost of Cash

Assume there is a disastrous shock that wipes out all cash and liquidates the firm. The Poisson shock N arrives with intensity $\delta > 0$. In addition, cash holdings grow at interest.

Continuation Value. By the martingale representation theorem, there exist processes β, η such that:

$$dU_t = rU_t dt - u(c_t)dt + \beta_t(-\rho r U_t)(dX_t - \mu dt) - \eta_t(-\rho r U_t)(dN_t - \delta dt).$$

The insider consumes according to $rU_t = u(c_t)$ so that:

$$dU_t = \beta_t(-\rho r U_t)(dX_t - \mu dt) - \eta_t(-\rho r U_t)(dN_t - \delta dt).$$

For

$$W(U) := \frac{-\ln(-\rho r U)}{\rho r}.$$

we can derive using Ito's Lemma:

$$dW_t = \frac{\rho r}{2}(\beta_t \sigma)^2 + \beta_t(dX_t - \mu dt) + \delta \eta_t - \frac{\ln(1 + \rho r \eta_t)}{\rho r} dN_t.$$

Define $Y_t = W_t - S_t$. In case of sudden liquidation with $dN_t = 1$, the insider looses Y_t so that:

$$Y_t = \frac{\ln(1 + \rho r \eta_t)}{\rho r},$$

which can be solved for:

$$\eta_t = \eta(Y_t) = \frac{\exp(\rho r Y_t) - 1}{\rho r}.$$

Note that $\eta'(Y) > 0$.

State Variables. Derive

$$dY_t = dW_t - dS_t = \frac{\rho r}{2}(\beta_t \sigma)^2 + \beta_t(dX_t - \mu dt) - rY_t dt - dw_t - Y_t dN_t + \eta_t \delta dt.$$

Cash holdings follows (2):

$$dM_t = rM_t dt + \sigma dZ_t - dDiv_t - dw_t - M_t dN_t,$$

so that

$$dC_t = rC_t dt - \delta \eta(Y_t) + \left(\mu - \frac{\rho r}{2}(\beta_t \sigma)^2\right) dt + (1 - \beta_t)\sigma dZ_t - dDiv_t - C_t dN_t.$$

Hence, the endogeneous carry cost of cash are given by $\delta \eta(Y)$ and increase in the extent of deferred compensation:

$$Y_t = \frac{\varphi_t C_t}{1 - \varphi_t}.$$

HJB equation. We obtain the HJB equation on $[0, \overline{C}]$:

$$(r+\delta)P(C) = \max_{\beta,\varphi\geq\lambda} \left\{ P'(C) \left[rC - \delta\eta\left(\frac{\varphi C}{1-\varphi}\right) - \frac{\rho r}{2}(\beta\sigma)^2 + \mu \right] + \frac{\sigma^2(1-\beta)^2}{2}P''(C) \right\}. \quad \text{(I.1)}$$

subject to $P'(\overline{C}) - 1 = P''(\overline{C})$. Clearly, as $\eta'(Y) > 0$, it is optimal to set $\varphi = \lambda$. As a result, the HJB equation is fairly similar to (23). Conclusively, we do not expect qualitatively different results if we endogenize the carry cost of cash.

I.2 Same interest for savings

Let us assume that the insider's savings also grow at the interest rate $r - \delta$ in that:

$$dS_t = (r-\delta)S_t dt - c_t dt + dw_t.$$

Consumption smoothing. The following Lemma characterizes the insider's consumption under this specification.

Lemma 17. *Let w an Ito process. Consider the problem*

$$U_t = \max_{\{c\}_{s\geq t}} \mathbb{E}_t\left(\int_t^\infty e^{-r(s-t)}u(c_s)ds\right)$$
$$\text{subject to } dS_t = (r-\delta)S_t dt - c_t dt + dw_t \quad \text{and} \quad \lim_{t\to\infty} e^{-rt}S_t = 0 \quad a.s.$$

The solution $\{c\}$ then satisfies $(r-\delta)U_t = u(c_t)$ for any $t \geq 0$.

Proof. By standard arguments (see e.g. Di Tella and Sannikov (2016) for this result in a dynamic contracing framework), the Euler equation reads

$$u'(c_{s'}) = \mathbb{E}_{s'}u_c(\tilde{c}_s)e^{-\delta(s-s')}$$

for any $t \leq s' < s$, which implies that discounted marginal utility $u'(c_s)e^{-\delta s}$ follows a martingale. Due to CARA preferences it follows that marginal utility $u'(\cdot)$ is proportional to flow utility $u(\cdot)$, so that $u(c_s)e^{-\delta s}$ is also a martingale. Hence,

$$\begin{aligned}
U_t &= \mathbb{E}_t\left(\int_t^\infty e^{-r(s-t)}u(c_s)ds\right) \\
&= \mathbb{E}\left(\int_t^\infty e^{-r(s-t)}e^{\delta s}e^{-\delta s}u(c_s)ds\right) \\
&= \int_t^\infty e^{-r(s-t)}e^{\delta s}\mathbb{E}\left(e^{-\delta s}u(c_s)\right)ds \\
&= \int_t^\infty e^{-r(s-t)}e^{\delta s}e^{-\delta t}u(c_t)ds \\
&= \frac{u(c_t)}{r-\delta} \\
\implies (r-\delta)U_t &= u(c_t),
\end{aligned}$$

which was to show. $\square$

Continuation Value. Regardless of the interest rate, (9) remains valid in that:

$$dU_t = rU_t dt - u(c_t)dt + \beta_t(-\rho r U_t)(dX_t - \mu dt),$$

which becomes by virtue of Lemma 17 under the optimal consumption c:

$$dU_t = \delta U_t dt + \beta_t(-\rho r U_t)(dX_t - \mu dt).$$

For

$$W(U) := \frac{-\ln(-\rho r U)}{\rho r}.$$

we can derive using Ito's Lemma:

$$dW_t = \frac{\rho r}{2}(\beta_t \sigma)^2 + \beta_t(dX_t - \mu dt) - \frac{\delta}{\rho r}.$$

State Variables. Derive

$$dY_t = dW_t - dS_t = \frac{\rho r}{2}(\beta_t \sigma)^2 + \beta_t(dX_t - \mu dt) - \frac{\delta}{\rho r} - (r-\delta)Y_t dt - dw_t.$$

Cash holdings follows (2):

$$dM_t = ((r-\delta)M_t + \mu)dt + \sigma dZ_t - dDiv_t - dw_t - db_t,$$

so that

$$dC_t = (r-\delta)C_t dt + \left(\mu + \frac{\delta}{\rho r} - \frac{\rho r}{2}(\beta_t\sigma)^2\right)dt + (1-\beta_t)\sigma dZ_t - dDiv_t.$$

HJB equation. Notably, dC_t does not depend on $\varphi = Y/M$ so that we may in optimum (without loss of generality) set $\varphi = \lambda$. We obtain the HJB equation on $[0, \overline{C}]$:

$$rP(C) = \max_{\beta \geq \lambda}\left\{P'(C)\left[(r-\delta)\,C - \frac{\rho r}{2}(\beta\sigma)^2 + \mu + \frac{\delta}{\rho r}\right] + \frac{\sigma^2(1-\beta)^2}{2}P''(C)\right\}. \tag{I.2}$$

subject to $P'(\overline{C}) - 1 = P''(\overline{C})$. The HJB equation is fairly similar to (23). Conclusively, we do not expect qualitatively different results if the insider earns interest at rate $r-\delta$ rather than r.

I.3 Determining the optimal refinancing policy

To start with, note the model solution is fully characterized by an equilibrium domain, which is fully described by the payout boundary $\overline{C}$, the value function P, the controls $\alpha, \beta, \gamma, \varphi$ on this domain as well as the refinancing policy C^*. We solve for the optimal refinancing policy iteratively. That is to say, we perform the following steps:

1. Solve the model without refinancing, i.e., for $\pi = 0$, which yields the solution triple $(P^0, \beta^0, \varphi_0, \overline{C}^0)$. Set $i \mapsto 1$. Set the default refinancing policy $C_0^* \mapsto 0$ and $(\alpha_0, \gamma_0) \mapsto (0, 0)$.

2. Solve the HJB-equation, taking the optimal refinancing policy $\alpha_i, , \beta_i, \gamma_i, \varphi_i$ and C_i^* as given. This yields the solution $(P, \overline{C}, \beta, \varphi)$.

3. Given $(P, \overline{C}, \beta, \varphi)$, calculate the optimal refinancing policy $(C^*, \alpha^*, \gamma^*)$.

4. Set $i \mapsto i+1$ and $(P_i, \alpha_i, \beta_i, \gamma_i, \varphi_i, \overline{C}_i, C_i^*) \mapsto (P, \alpha^*, \beta, \gamma^*, \varphi, \overline{C}, C^*)$

5. If

$$||P_i - P_{i-1}||_\infty^{[0,\infty)} < \epsilon$$

   for some tolerance $\epsilon > 0$, the solution is obtained. Otherwise go back to step 2. Here, $||\cdot||_\infty^{[a,b]}$ is the supremum norm on the interval $[a, b]$.